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DISCRETE MATHEMATICS IN RELATION TO COMPUTER SCIENCE

On the Approximation of the Resource Equivalences in Petri Nets with the Invisible Transitions

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Two resources (submarkings) are called similar if in any marking any one of them can be replaced by another one without affecting the observable behavior of the net (regarding marking bisimulation). It is known that resource similarity is undecidable for general labelled Petri nets. In this paper we study the properties of the resource similarity and resource bisimulation (a subset of complete similarity relation closed under transition firing) in Petri nets with invisible transitions (where some transitions may be labelled with an invisible label (τ) that makes their firings unobservable for an external observer). It is shown that for a proper subclass (p-saturated nets) the resource bisimulation can be effectively checked. For a general class of Petri net with invisible transitions it is possible to construct a sequence of so-called (n, m)-equivalences approximating the largest τ-bisimulation of resources.

Keywords: resource, equivalence, Petri nets, invisible transitions, approximation.

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DISCRETE MATHEMATICS IN RELATION TO COMPUTER SCIENCE

Аппроксимация ресурсных эквивалентностей в сетях Петри с невидимыми переходами

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Два ресурса (подразметки) называются подобными, если в любой разметке любой из них может быть заменен другим, и при этом наблюдаемое поведение сети не изменится (относительно бисимуляции разметок). Известно, что подобие ресурсов неразрешимо для обыкновенных сетей Петри. В этой статье мы изучаем свойства подобия ресурсов и бисимуляции ресурсов (подмножество отношения подобия, замкнутое по срабатыванию переходов) в сетях Петри с невидимыми переходами (где некоторые переходы могут быть помечены специальной меткой (τ), что делает их срабатывания невидимыми для внешнего наблюдателя). Показано, что для собственного подкласса (p-насыщенных сетей) бисимуляция ресурсов может быть эффективно проверена. Для общего класса сетей Петри с невидимыми переходами можно построить последовательность так называемых (n, m)-эквивалентностей, аппроксимирующую наибольшую τ-бисимуляцию ресурсов.

Ключевые слова: ресурс, эквивалентность, сети Петри, невидимые переходы, аппроксимация.

ИНФОРМАЦИЯ ОБ АВТОРАХ

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1. Introduction

In this paper the behavior of Petri nets is investigated from the standpoint of bisimulation equivalence. The fundamental notion of bisimulation was introduced by R. Milner [1] and D. Park [2]. Two markings of a Petri net are called bisimilar if the choice of each of them as an initial marking gives the same visible behavior of the net. In [3] P. Jančar proved that bisimulation equivalence of markings is undecidable for a general Petri net.

In [4] C. Autant et al. introduced a notion of place bisimulation — a decidable bisimulation-induced equivalence on the finite set of places, that allows to find out some non-trivial behavior-preserving net reductions. This relation and its applications were studied in [4–6].

The notion of resource similarity was introduced in [7]. In general a resource is a submarking. Two resources are similar if, having replaced one resource in any marking by another, we obtain the same observed behavior of the net. Resource bisimulation is a particular case of similarity that is closed under transition firing. Place bisimulation is a proper subset of resource bisimulation. Note that, unlike the place bisimulation [4], resource similarity and bisimulation are defined on the infinite set (of resources/submarkings).

Resource similarity and its modifications where studied in [7–9]. In particular it was proven that resource similarity is undecidable. However, it was shown that resource bisimulation can be effectively approximated and used as a basis of net reductions and adaptive control. For an overview, see [10].

This article is an extended version of the workshop report [11]. We consider an important generalization of labelled Petri nets, where some transitions may be labelled with an invisible label (tau), that makes their firings unobservable for an external observer. Quite often when analyzing the system there is a need to abstract from the excessive information about its behavior. For example, it is convenient to hide all transitions, corresponding to the internal actions of the system. The information obtained in this case can be useful, in particular, to detect additional properties of the system in terms of its interaction with the environment.

Place bisimulations in Petri nets with invisible transitions were studied by C. Autant et al. in [5]. It was shown that unlabelled sequences of steps significantly complicate the calculations. However, there are specific nontrivial subclasses of Petri nets with invisible transitions, that have some nice properties w.r.t. place bisimulation.

In this paper we apply a similar approach to the resource equivalences. It is shown that resource bisimulations can be effectively computed in some non-trivial subclasses of nets with invisible transitions.

A class of p-saturated nets is studied. In p-saturated nets the firing of any sequence of transitions with at most one visible label can be simulated by a simultaneous (independent) firing of a certain set of transitions with the same label (called parallel step). In p-saturated Petri nets τ-bisimulation coincides with the so-called τp-bisimulation [5], that takes into account parallel steps instead of transition sequences.

It is shown that in the class of p-saturated nets the weak transfer property of resource τp-bisimulation can be effectively checked. Moreover, we can underapproximate the largest τp-bisimulation by a parameterized algorithm.

It is shown that for a general class of Petri net with invisible transitions it is possible to construct a sequence of so-called (n, m)-equivalences, approximating the largest τ-bisimulation of resources.

The paper is organized as follows. Section 2 contains basic definitions. Specifically, in Subsection 2.1 we give some technical notions and lemmata on the properties of additively-transitively closed relations on multisets. Subsection 2.2 contains definitions of Petri nets and bisimulations. Subsections 2.3 and 2.4 give a short review on Petri net resources and resource equivalences (similarity and bisimulation). Section 3 deals with invisible transitions. In Subsections 3.1 and 3.2 we define the τ-generalizations of resource equivalences and study their properties. It is shown that the straightforward method of bisimulation checking with a weak transfer property is not applicable here. In Section 4 we study the subclass of p-saturated nets and the corresponding notion of τp-bisimulation. In Subsection 4.3 we present an algorithm, computing the parameterized underapproximation of largest τp-bisimulation. Section 5 is devoted to the general case of Petri nets with invisible transitions. A parameterized approximation procedure for resource bisimulation is defined and studied. Section 6 contains some conclusions.

2. Preliminaries

2.1. Relations on multisets

Denote by ε an empty sequence. Let X and Y be two sets. Let $\sigma \in X^*$ be a sequence over X. Denote by $\sigma_{|Y}$ a *projection* of σ onto Y such that for an empty sequence $\sigma = \varepsilon$ we have $\sigma_{|Y} =_{def} \varepsilon$ and for a non-empty sequence $\sigma = a\delta$ with $a \in X$ and $\delta \in X^*$ we have $\sigma_{|Y} =_{def} a\delta_{|Y}$ for $a \in Y$ and $\sigma_{|Y} =_{def} \delta_{|Y}$ for $a \notin Y$.

A *multiset* M over a set X is a mapping $M : X \longrightarrow Nat$, where Nat is the set of natural numbers (including zero), i.e. a multiset may contain several copies of the same element.

Size of a multiset is defined as follows: $|M| = \sum_{x\in X} M(x)$. A multiset M is finite if a set $\{x \in X \mid M(x) > 0\}$ is finite. By $\mathcal{M}(X)$ we denote the set of all finite multisets over X. An empty multiset is denoted by $\emptyset$.

The operations and relations of set theory are naturally extended to finite multisets. Let $M_1, M_2, M_3 \in \mathcal{M}(X)$. Then:

- $M_1 = M_2 \iff_{def} \forall x \in X\ M_1(x) = M_2(x)$;
- $M_1 \subseteq M_2 \iff_{def} \forall x \in X\ M_1(x) \leq M_2(x)$;
- $M_1 \subset M_2 \iff_{def} M_1 \subseteq M_2 \ \wedge\ \exists x \in X\ M_1(x) \leq M_2(x)$;
- $M_1 = M_2 + M_3 \iff_{def} \forall x \in X\ M_1(x) = M_2(x) + M_3(x)$;
- $M_1 = M_2 \cap M_3 \iff_{def} \forall x \in X\ M_1(x) = min\{M_2(x), M_3(x)\}$;
- $M_1 = M_2 - M_3 \iff_{def} \forall x \in X\ M_1(x) = max\{0, M_2(x) - M_3(x)\}$;
- $M_1 = kM_2,\ k \in Nat \iff_{def} \forall x \in X\ M_1(x) = kM_2(x)$;
- $M_1 = (M_2)_{|Y},\ Y \subseteq X \iff_{def} \forall x \in X\ M_1(x) = M_2(x)$ for $x \in Y$ and $M_1(x) = 0$ otherwise.

Non-negative integer vectors are often used to encode multisets. Actually, the set of all multisets over finite X is a homomorphic image of $Nat^{|X|}$.

A binary relation $B \subseteq Nat^k \times Nat^k$ is a congruence if it is an equivalence relation and whenever $(v, w) \in B$ then $(v + u, w + u) \in B$ (here '+' denotes coordinate-wise addition).[1] It was proven by L. Redei [12] that every congruence on Nat^k is generated by a finite set of pairs. Later P. Jančar [3] and J. Hirshfeld [13] presented a shorter proof and also showed that every congruence on Nat^k is a semilinear relation, i.e. it is a finite union of linear sets.

Let B^{AT} denote the additive-transitive closure (AT-closure) of the relation $B \subseteq \mathcal{M}(X)\times\mathcal{M}(X)$ (the minimal congruence, containing B).

Let $B \subseteq \mathcal{M}(X) \times \mathcal{M}(X)$ be a binary relation on multisets. A relation B' is called an *AT-basis* of B iff $(B')^{AT} = B^{AT}$. An AT-basis B' is called *minimal* iff there is no $B'' \subset B'$ such that $(B'')^{AT} = B^{AT}$.

Now we construct a special kind of minimal AT-basis for B. Define a partial order $\sqsubseteq$ on the set $B \subseteq \mathcal{M}(X) \times \mathcal{M}(X)$ of pairs of multisets as follows:

1. For loop (i.e. reflexive) pairs let
$$(r_1, r_1) \sqsubseteq (r_2, r_2) \overset{def}{\iff} r_1 \subseteq r_2;$$
2. For two non-loop pairs, the maximal loop constituents and the addend pairs of nonintersecting multisets are compared separately
$$(r_1 + o_1, r_1 + o_1') \sqsubseteq (r_2 + o_2, r_2 + o_2') \overset{def}{\iff}$$
$$\overset{def}{\iff} o_1 \cap o_1' = \emptyset \ \&\ o_2 \cap o_2' = \emptyset \ \&\ r_1 \subseteq r_2 \ \&\ o_1 \subseteq o_2 \ \&\ o_1' \subseteq o_2'.$$

[1] Note that it can be easily seen that if B is a congruence and $(v, w), (u, x) \in B$ then also $(v + u, w + x) \in B$.

3. a loop pair and a non-loop pair are always incomparable.
Let B_s denote the set of all minimal (with respect to $\sqsubseteq$) elements of B^{AT}.

Theorem 1. *[8] Let $B \subseteq \mathcal{M}(X) \times \mathcal{M}(X)$ be a symmetric and reflexive relation. Then B_s is an AT-basis of B and B_s is finite.*

We call B_s the *ground basis* of B. Obviously, it is finite.

There is also a useful

Lemma 1. *[8] Let $B \subseteq \mathcal{M}(X) \times \mathcal{M}(X)$ be a symmetric and reflexive relation, $(r, s) \in B^{AT}$. Then there exists a finite chain of pairs*

$$(r, a_1), (a_1, a_2), \dots, (a_{k-1}, a_k), (a_k, s) \in (B_s)^A,$$

where $(B_s)^A$ is the additive closure of B_s.

2.2. Labelled Petri nets and bisimulations

Let P and T be disjoint sets of *places* and *transitions* and let $F : (P \times T) \cup (T \times P) \longrightarrow Nat$. Then $N = (P, T, F)$ is a *Petri net.* a *marking* in a Petri net is a function $M : P \longrightarrow Nat$, mapping each place to some natural number (possibly zero). Thus a marking may be considered as a multiset over the set of places. Pictorially, P-elements are represented by circles, T-elements by boxes, and the flow relation F by directed arcs. Places may carry tokens represented by filled circles. a current marking M is designated by putting $M(p)$ tokens into each place $p \in P$. Tokens residing in a place are often interpreted as resources of some type consumed or produced by a transition firing. a marked Petri net (N, M_0) is a Petri net N together with a given initial marking M_0.

For a transition $t \in T$ the *preset* ${}^\bullet t$ and the *postset* $t^\bullet$ are defined as the multisets over P such that ${}^\bullet t(p) = F(p, t)$ and $t^\bullet(p) = F(t, p)$ for each $p \in P$.

A transition $t \in T$ is *enabled* in a marking M iff ${}^\bullet(t) \subseteq M$. An enabled transition t may *fire* yielding a new marking $M' =_{\text{def}} M - {}^\bullet t + t^\bullet$, i.e. $M'(p) = M(p) - F(p, t) + F(t, p)$ for each $p \in P$ (denoted $M \xrightarrow{t} M'$).

Let $\sigma \in T^*$ be a sequence of transition (possibly empty), $t \in T$ – a transition. The pre- and postcondition for a non-empty sequence are defined inductively:

$${}^\bullet(t\sigma) =_{def} {}^\bullet t + ({}^\bullet\sigma - t^\bullet), \qquad (\sigma t)^\bullet =_{def} t^\bullet + (\sigma^\bullet - {}^\bullet t).$$

A sequence $\sigma \in T^*$ is *enabled* in M iff ${}^\bullet\sigma \subseteq M$. An enabled sequence may *fire* yielding a new marking $M' =_{\text{def}} M - {}^\bullet\sigma + \sigma^\bullet$ (denoted $M \xrightarrow{\sigma} M'$).

A multiset of transitions may *fire in parallel* (concurrently), if there are enough tokens for all of them. a transition may fire in parallel with itself. The concurrent firing of a multiset of transitions is called *a parallel step.* The pre- and postcondition for a multiset of transitions $U \in \mathcal{M}(T)$ are:

$${}^\bullet U =_{def} \sum_{t \in T} U(t) \times {}^\bullet t, \qquad U^\bullet =_{def} \sum_{t \in T} U(t) \times t^\bullet.$$

A parallel step $U \in \mathcal{M}(T)$ is *enabled* in M iff ${}^\bullet U \subseteq M$. An enabled parallel step may *fire* yielding a new marking $M' =_{\text{def}} M - {}^\bullet U + U^\bullet$ (denoted $M \xrightarrow{U} M'$).

Obviously, we have ${}^\bullet(U + W) = {}^\bullet U + {}^\bullet W, (U + W)^\bullet = U^\bullet + W^\bullet$.

To observe the net behavior transitions are labelled by special labels representing observable actions or events. Let Act be a set of action names. A *labelled Petri net* is a tuple $N = (P, T, F, l)$, where (P, T, F) is a Petri net and $l : T \longrightarrow Act$ is a labelling function. It can be generalized to non-empty sequences:

$$\text{for } \alpha \in T^* \text{ s.t. } \alpha = t\beta \text{ with } t \in T \text{ and } \beta \in T^* \text{ we have } l(\alpha) =_{def} l(t)l(\beta).$$

And also to multisets of transitions (note that in this case labels are not sequences but multisets of action names):

$$\text{for } U \in \mathcal{M}(T) \qquad l(U) =_{def} \textstyle\sum_{t \in T} U(t) \times l(t).$$

Let $N = (P, T, F, l)$ be a labelled Petri net. We say that a relation $B \subseteq \mathcal{M}(P) \times \mathcal{M}(P)$ conforms to the *transfer property* iff for all $(M_1, M_2) \in B$ and for every step $t \in T$, s.t. $M_1 \xrightarrow{t} M_1'$, there exists an imitating step $u \in T$, s.t. $l(t) = l(u)$, $M_2 \xrightarrow{u} M_2'$ and $(M_1', M_2') \in B$.

A relation B is called a *marking bisimulation*, if both B and B^{-1} conform to the transfer property.

It is known that a union of two marking bisimulations is a marking bisimulation. Hence for every labelled Petri net there exists the largest marking bisimulation (a union of all bisimulations; denoted by $\sim$) and this bisimulation is an equivalence. It was proved by P. Jančar [3], that the marking bisimulation is undecidable for Petri nets. More precisely, it is undecidable whether two markings (of the same net) are marking bisimilar, even if restricted to nets with only two unbounded places.

2.3. Resource similarity

Informally, resources are parts of markings which may or may not provide some particular kind of observable net behavior.

Definition 1. *[8] Let $N = (P, T, F, l)$ be a labelled Petri net. a* resource $R \in \mathcal{M}(P)$ *in a Petri net N is a multiset over the set of places P.*

Resources r and s in N are called similar *(denoted $r \approx s$) iff for every marking $R \in \mathcal{M}(P)$, $r \subseteq R$ implies $R \sim R - r + s$.*

Thus if two resources are similar, then in every marking each of these resources can be replaced by the other without changing the observable behavior of the system. Here we consider the observability modulo action names: the external observer can see events (labels of fired transitions) but cannot distinguish local states (tokens). Some examples of similar resources are shown in Fig. 1.

Figure a) shows a Petri net containing two transitions labeled with the same label a and leading to the same marking p_3. Here the resources p_1 and p_2 are similar, as they lead to a completely identical observable behavior — action a producing a single token in p_3. Moreover, all the resources containing the same number of tokens in p_1 and p_2 are similar.

Figure b) shows a simple net consisting of a single transition. In this case the resource p_2 is similar to an empty resource, since it does not affect the behavior of the net (the place p_2 is redundant).

Figure c) depicts a cycle consisting of one transition and one place. Note that the set of markings of this net can be divided into two disjoint subsets — empty marking and all the others. With empty marking, the transition can not fire, for all others — it can fire any number of times. Note that for this net the largest marking bisimulation and the resource similarity coincide. Also note that marking bisimulation takes into account only steps made of single transitions hence no auto-concurrency can be considered here.

Figure d) shows a more complex situation. We have $p_1 \approx p_2 + p_3$, that is, replacing one token in p_1 by two tokens (one in p_2 and one in p_3) does not affect the observable behavior of the net as a whole.

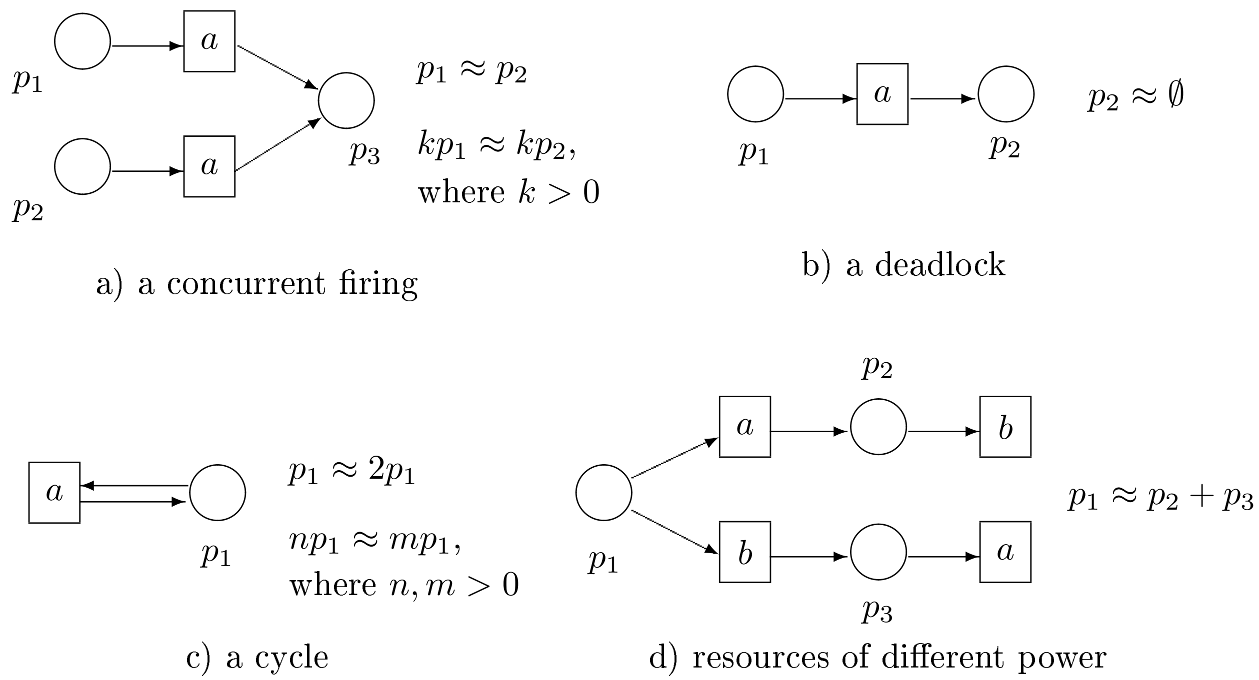

Fig. 1. Examples of similar resources

Рис. 1. Пример подобных ресурсов

The similarity relation is an equivalence [8]. Moreover, it is a congruence w.r.t. multiset addition:

Proposition 1. *[8] Let $N = (P, T, F, l)$ be a labelled Petri net, let r, s, u, v be resources of the net N. Then $r \approx s \;\&\; u \approx v \implies r + u \approx s + v$.*

Hence it has a finite ground basis. Unfortunately, from the undecidability of a stronger relation of place fusion [6] we get

Theorem 2. *[8] The resource similarity is undecidable for labelled Petri nets.*

2.4. Resource bisimulation

The resource similarity is quite fundamental, but the undecidability makes it not very useful in practice. So we studied a number of other non-trivial finitely-based resource equivalence relations, retaining the observable system's behavior. The most interesting of them is a resource bisimulation:

Definition 2. *[8] Let $N = (P, T, F, l)$ be a labelled Petri net. An equivalence relation $B \subseteq \mathcal{M}(P) \times \mathcal{M}(P)$ is called a resource bisimulation if B^{AT} is a marking bisimulation.*

Note that an AT-closure of a resource similarity relation is not necessarily a marking bisimulation (it is still an open question [10]). However, we already know that each resource bisimulation B is a subset of resource similarity relation ($\approx$). The following theorem states this and some other important properties of resource bisimulations.

Theorem 3. *[8] Let $N = (P, T, F, l)$ be a labelled Petri net. Then*

1. *if $B \subseteq \mathcal{M}(P) \times \mathcal{M}(P)$ is a resource bisimulation and $(r_1, r_2) \in B$ then $r_1 \approx r_2$;*
2. *if B_1, B_2 are resource bisimulations for N then $B_1 \cup B_2$ is a resource bisimulation for N;*
3. *for any N there exists the largest resource bisimulation (denoted by $B(N)$), and it is an equivalence.*

Therefore $B(N)$ (as well as any other resource bisimulation) also has a finite ground basis.

The AT-closure of a resource bisimulation is a marking bisimulation, and hence, it conforms to the transfer property. Resource bisimulations satisfy a weak variant of the transfer property, considering only minimal pairs of markings that contain the corresponding resources and enable the corresponding transitions.

We say that a relation $B \subseteq \mathcal{M}(P) \times \mathcal{M}(P)$ conforms to *the weak transfer property* if for all $(r, s) \in B$, for each $t \in T$, such that ${}^\bullet t \cap r \neq \emptyset$, there exists an imitating transition $u \in T$, such that $l(t) = l(u)$ and, writing M_1 for ${}^\bullet t \cup r$ and M_2 for ${}^\bullet t - r + s$, we have $M_1 \xrightarrow{t} M_1{}'$ and $M_2 \xrightarrow{u} M_2{}'$ with $(M_1', M_2') \in B^{AT}$.

Theorem 4. *[8] Let $N = (P, T, F, l)$ be a labelled Petri net. A relation $B \subseteq \mathcal{M}(P)\times\mathcal{M}(P)$ is a resource bisimulation iff B is an equivalence relation and it conforms to the weak transfer property.*

Due to this theorem to check whether a given finite relation B is a resource bisimulation, one needs to verify the weak transfer property for only a finite number of pairs of resources. In [8] we have shown that the largest resource bisimulation for resources with a bounded number of tokens can be effectively constructed (more precisely, it requires $O(\max\{|P|\,\mathcal{R}^9, |T|^2|P|\,\mathcal{R}^7\})$ steps, where $\mathcal{R}$ is the number of resources in the consideration).

3. Petri nets with invisible transitions

In this section we investigate the possibilities of effectively constructing bisimulation-preserving relations for an extended class of systems — Petri nets with invisible transitions.

To distinguish visible and invisible transitions, a special τ symbol is added to the set of labels: $Act_\tau = Act \cup \{\tau\}$.

Definition 3. *A* labelled Petri net with invisible transitions *is a tuple $N = (P, T, F, l)$, where (P, T, F) is a Petri net and $l : T \longrightarrow Act_\tau$ is an extended labelling function.*

Let $\sigma, \sigma' \in (Act_\tau)^*$ be sequences of action labels. Denote $\sigma =_\tau \sigma' \iff_{def} \sigma_{|Act} = \sigma'_{|Act}$ ("equal modulo τ"). For example, "$\tau\tau a\tau$" $=_\tau$ "a".

Similarly, let $U, U' \in \mathcal{M}(Act_\tau)$ be multisets of action labels. Denote $U =_\tau U' \iff_{def} U_{|Act} = U'_{|Act}$. For example, $\{a, \tau, a, b, \tau\} =_\tau \{a, a, b\}$.

3.1. τ-bisimulation

Let $N = (P, T, F, l)$ be a labelled Petri net with invisible transitions. We say that a relation $B \subseteq \mathcal{M}(P) \times \mathcal{M}(P)$ conforms to the *τ-transfer property* iff for all $(M_1, M_2) \in B$ and for every step $t \in T$, s.t. $M_1 \xrightarrow{t} M_1'$, there exists an imitating sequence of steps $\sigma \in T^*$ s.t. $l(t) =_\tau l(\sigma)$, $M_2 \xrightarrow{\sigma} M_2'$ and $(M_1', M_2') \in B$.

A relation B is called a *marking τ-bisimulation*, if both B and B^{-1} conform to the τ-transfer property. The largest τ-bisimulation is denoted by $\sim_\tau$.

Marking bisimulation is a special case of marking τ-bisimulation (for nets with no τ-s). It is a stronger relation. Consider as an example the net depicted in Fig. 2. Markings p_1 and p_2 are not bisimilar, because at p_2 no transition with label a is active. But they are τ-bisimilar, because the invisible firing of t_2 changes the marking from p_2 to p_1.

In particular, this implies the undecidability of marking τ-bisimulation in Petri nets with invisible transitions [3].

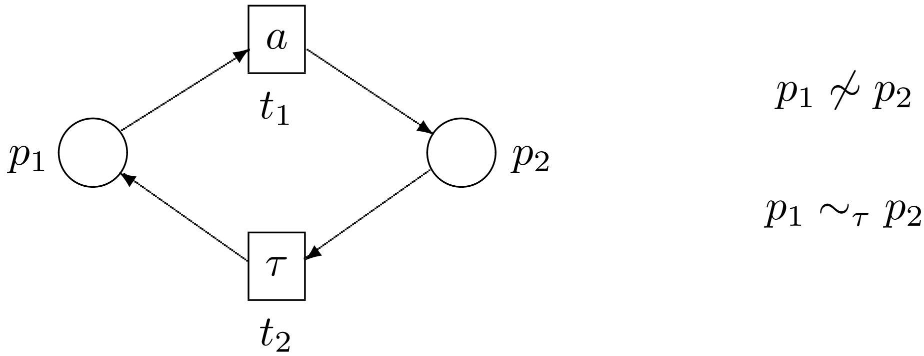

Fig. 2. τ-bisimulation is weaker than bisimulation

Рис. 2. τ-бисимуляция слабее, чем обычная бисимуляция

3.2. Resource similarity and bisimulation

The definition of resource similarity can be naturally generalized to the case of nets with invisible transitions:

Definition 4. *Let $N = (P, T, F, l)$ be a labelled Petri net with invisible transitions. Resources r and s are called* τ-similar *(denoted $r \approx_\tau s$) iff for every marking R, $r \subseteq R$ implies $R \sim_\tau R - r + s$.*

We can show that resource τ-similarity has all basic properties of resource similarity:

Proposition 2.
1. *Resource τ-similarity is closed under addition and is transitive; hence it has finite AT-basis.*
2. *Resource τ-similarity is undecidable.*

Proof. 1) From the definitions.

2) From Th. 2 (note that τ-similarity is a generalization of basic resource similarity). □

The definition of resource bisimulation also can be easily generalized:

Definition 5. *Let $N = (P, T, F, l)$ be a labelled Petri net with invisible transitions. An equivalence relation $B \subseteq \mathcal{M}(P) \times \mathcal{M}(P)$ is called a* resource τ-bisimulation *if B^{AT} is a marking τ-bisimulation.*

Proposition 3. *Let $N = (P, T, F, l)$ be a labelled Petri net with invisible transitions. Then*

1. if $B \subseteq \mathcal{M}(P) \times \mathcal{M}(P)$ is a resource τ-bisimulation and $(r_1, r_2) \in B$ then $r_1 \approx_\tau r_2$;
2. if $B_1, B_2 \subseteq \mathcal{M}(P) \times \mathcal{M}(P)$ are resource τ-bisimulations then $B_1 \cup B_2$ is a resource τ-bisimulation;
3. for any N there exists the largest resource τ-bisimulation (denoted by $B_\tau(N)$), and it is an equivalence.

Proof. 1) We need to prove that $r_1 \approx_\tau r_2$: for any $R \in \mathcal{M}(P)$ s.t. $r_1 \subseteq R$ we have $R \sim_\tau R - r_1 + r_2$.

Denote $r' = R - r_1$. The pair $(R, R - r_1 + r_2)$ can be represented as $(r_1 + r', r_2 + r')$, therefore it belongs to B^{AT}. Since B is a resource τ-bisimulation, B^{AT} is a marking τ-bisimulation, and hence it is a subset of a largest marking τ-bisimulation $(\sim_\tau)$. So, we obtained $R \sim_\tau R - r_1 + r_2$.

2) Denote $B = B_1 \cup B_2$. We need to prove that B is a resource τ-bisimulation: for any $(M_1, M_2) \in B^{AT}$ we have $M_1 \sim_\tau M_2$.

Consider the structure of (M_1, M_2). From Lm. 1 we have

$$(M_1, a_1), (a_1, a_2), \dots, (a_{k-1}, a_k), (a_k, M_2) \in (B_s)^A$$

for some finite k, where $(B_s)^A$ is the additive closure of B_s.

It can be easily seen that $B_s \subseteq (B_1)_s \cup (B_2)_s$, hence for any $(X, Y) \in (B_s)^A$ we have $X = X_1 + X_2$, $Y = Y_1 + Y_2$ s.t. $(X_1, Y_1) \in ((B_1)_s)^A$ and $(X_2, Y_2) \in ((B_2)_s)^A$.

From the reflexivity of B_1 and B_2 and additive closureness of $(B_1)^{AT}$ and $(B_2)^{AT}$ we have $(X_1+X_2, Y_1+X_2) \in (B_1)^{AT}$ and $(Y_1 + X_2, Y_1 + Y_2) \in (B_2)^{AT}$. Both B_1 and B_2 are resource τ-bisimulations, so $(B_1)^{AT}$ and $(B_2)^{AT}$ are marking τ-bisimulations. Therefore they are both contained in the largest τ-bisimulation ($\sim_\tau$), so we have $X_1 + X_2 \sim_\tau Y_1 + X_2$ and $Y_1 + X_2 \sim_\tau Y_1 + Y_2$. The bisimulation is transitive, hence $X_1 + X_2 \sim_\tau Y_1 + Y_2$.

So for any $(X, Y) \in (B_s)^A$ we have $X \sim_\tau Y$. Applying this reasoning to the pairs in our chain, we obtain $M_1 \sim_\tau a_1, a_1 \sim_\tau a_2, \dots, a_{k-1} \sim_\tau a_k, a_k \sim_\tau M_2$. Hence, $M_1 \sim_\tau M_2$.

3) The third statement is an immediate corollary of the second one. The largest resource τ-bisimulation can be constructed as the union of all resource τ-bisimulations for N. □

Definition 6. *We say that a relation $B \subseteq \mathcal{M}(P) \times \mathcal{M}(P)$ conforms to the* weak τ-transfer property *if for all $(r, s) \in B$, $t \in T$ s.t. ${}^\bullet t \cap r \neq \emptyset$, there exists an imitating sequence of transitions $\sigma \in T^*$ s.t. $l(t) =_\tau l(\sigma)$ and, denoting $M_1 = {}^\bullet t \cup r$ and $M_2 = {}^\bullet t - r + s$, we have $M_1 \xrightarrow{t} M_1{}'$ and $M_2 \xrightarrow{\sigma} M_2{}'$ with $(M_1', M_2') \in B^{AT}$.*

Th. 4 in the case of Petri nets with invisible transitions works only in one direction:

Proposition 4. *If the relation conforms to the τ-transfer property then it conforms to the weak τ-transfer property; there exist relations, conforming to the weak τ-transfer property and not conforming to the τ-transfer property.*

Proof. ($\Longrightarrow$) Since the weak τ-transfer property is the τ-transfer property for a bounded (finite) subset of pairs of resources.

($\not\Longleftarrow$) Consider the net depicted in Fig. 3 (this example is taken from [5]) and a relation

$$B = Id(P) \cup \{(p_1, p_2), (p_2, p_1), (p_3, p_4), (p_4, p_3)\},$$

where $Id(P)$ is an identity relation s.t. $\forall x, y \in P \quad (x, y) \in Id(P) \Longleftrightarrow x = y$.

B conforms to the weak τ-transfer property. At the same time B is not a resource τ-bisimulation. Consider markings $M_1 = p_1 + p_3$ and $M_2 = p_2 + p_4$. The pair (M_1, M_2) belongs to the relation B^{AT}, but the markings are not bisimilar, because an action a is possible at M_2 (transition t_3) and is impossible at M_1. □

Hence the weak τ-transfer property can not be used to construct bisimulation. In the case of systems with invisible transitions it is even more important to strengthen the considered relations and/or to restrict the considered class of Petri nets.

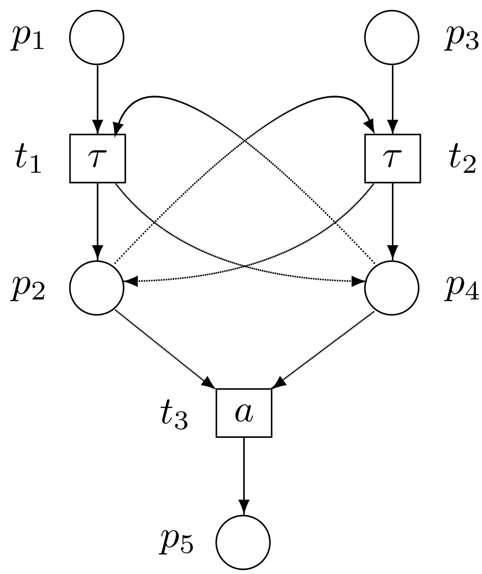

Fig. 3. Th. 4 does not hold for Petri nets with invisible transitions

Рис. 3. Теорема 4 не выполняется для сетей Петри с невидимыми переходами

4. Underapproximations of τ-similarity in saturated nets

4.1. Saturated nets

There exists a wide and important subclass of Petri nets with invisible transitions for which resource τ-bisimulation can be constructed using weak transfer property — so-called "p-saturated nets". In p-saturated nets [5] the firing of any sequence of transitions with at most one visible label can be simulated by a simultaneous (independent) firing of a certain set of transitions with the same label (called "parallel step").

Denote the set of non-empty transition sequences with at most one visible label:

$$T^{\times} =_{def} \{\sigma \in T^{*} \mid l(\sigma) \in Act_{\tau}\}.$$

Definition 7. *A labelled Petri net with invisible transitions $N = (P, T, F, l)$ is called p-*saturated *(or simply* saturated*), if for any sequence of transitions $\sigma \in T^{\times}$ there exists a parallel step $U \in \mathcal{M}(T)$ s.t. ${}^{\bullet}U = {}^{\bullet}\sigma$, $U^{\bullet} = \sigma^{\bullet}$ and, denoting by U_{σ} the multiset of transitions, participating in σ, we have $l(U) =_{\tau} l(U_{\sigma})$.*

In addition to saturated nets, there is an even broader class of *saturable* Petri nets. These are nets that can be transformed into saturated by adding a finite number of transitions while preserving the behavior of the net (in the sense of τ-bisimilarity). In Fig. 4 a saturated net is shown, obtained by adding the transition t_3 to the unsaturated net.

It is known [5] that a net is p-saturated iff it is $2p$-saturated, i.e. all sequences of length 2 are saturated by parallel steps.

Not all nets are saturable [5]. An example is given in Fig. 5. Here all transition sequences has the same precondition (a single token in the upper place) and different postconditions. So there is an infinite set of different transition sequences with different postconditions. On the other hand, the structure of the net also implies that all possible parallel steps with the same precondition (a single token in the upper place) would necessarily contain a single transition. Hence the number of different imitating parallel steps is always finite and equal to the number of existing transition. The saturation would not help, because it can not introduce an infinite number of new transitions.

It is also easy to see that the net is saturable iff its "invisible subnet" is saturable (an invisible subnet is a net, obtained by removing all visible transitions).

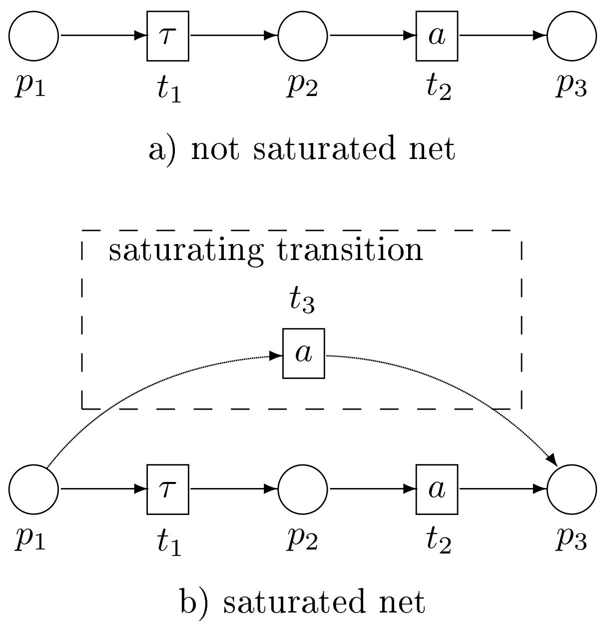

Fig. 4. An example of net saturation

Рис. 4. Пример насыщения сети

4.2. τp-bisimulation

In [5] an equivalence stronger than τ-bisimulation was defined, called τp-bisimulation of markings. The transition in this case is modeled not by a sequence of transitions, but by a parallel step.

Definition 8. *[5] Let $N = (P, T, F, l)$ be a labelled Petri net with invisible transitions. We say that a relation $B \subseteq \mathcal{M}(P) \times \mathcal{M}(P)$ conforms to the τp-*transfer property *if for all $(M_1, M_2) \in B$ and for each $t \in T$ s.t. $M_1 \xrightarrow{t} M_1'$, there exists an imitating parallel step $U \in \mathcal{M}(T)$ s.t. $\{l(t)\} =_\tau l(U)$, $M_2 \xrightarrow{U} M_2'$ and $(M_1', M_2') \in B$.*

Definition 9. *[5] A relation B is called a* marking τp-bisimulation, *if both B and B^{-1} conform to the τp-transfer property.*

It is known [5] that for any net there exists the largest τp-bisimulation (denoted by $\sim_{\tau p}$).
In saturated Petri nets τp-bisimulation coincides with τ-bisimulation [5]:

$$M_1 \sim_{\tau p} M_2 \quad \Longleftrightarrow \quad M_1 \sim_\tau M_2.$$

Now we are ready to define a resource τp-similarity:

Definition 10. *Let $N = (P, T, F, l)$ be a saturated labelled Petri net with invisible transitions. Resources r and s are called* τp-similar *(denoted $r \approx_{\tau p} s$) iff for every marking R, $r \subseteq R$ implies $R \sim_{\tau p} R - r + s$.*

From the equality of $\sim_{\tau p}$ and $\sim_\tau$ in saturated nets we immediately have:

Corollary 1. *Let $N = (P, T, F, l)$ be a saturated labelled Petri net with invisible transitions, $r, s \in \mathcal{M}(P)$. Then*

$$r \approx_{\tau p} s \quad \Longleftrightarrow \quad r \approx_\tau s.$$

So, in saturated nets it is sufficient to look for τp-similarities.

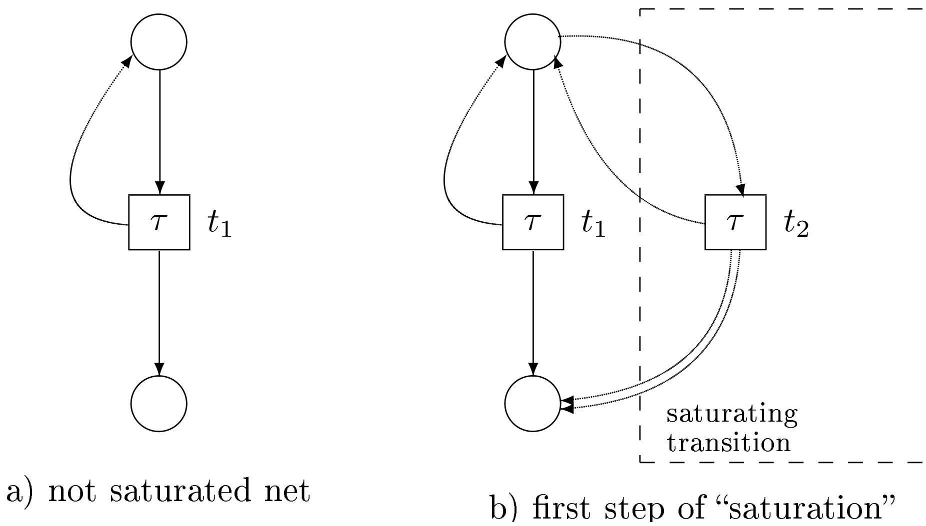

Fig. 5. Not saturable net **Рис. 5.** Не насыщаемая сеть

Definition 11. *Let $N = (P, T, F, l)$ be a saturated labelled Petri net with invisible transitions. An equivalence relation $B \subseteq \mathcal{M}(P) \times \mathcal{M}(P)$ is called a* resource τp-bisimulation *if B^{AT} is a marking τp-bisimulation.*

In the case of τp-relations all basic properties also hold:

Proposition 5. *1. Resource τp-similarity is closed under addition and transitivity; so it has finite AT-basis.*
2. Resource τp-similarity is undecidable.
3. If $B \subseteq \mathcal{M}(P) \times \mathcal{M}(P)$ is a resource τp-bisimulation and $(r_1, r_2) \in B$ then $r_1 \approx_{\tau p} r_2$.
4. If $B_1, B_2 \subseteq \mathcal{M}(P) \times \mathcal{M}(P)$ are resource τp-bisimulations then $B_1 \cup B_2$ is a resource τp-bisimulation;
5. For any N there exists the largest resource τp-bisimulation (denoted by $B_{\tau p}(N)$), and it is an equivalence.

Proof. 1) Immediately from the definition of resource τp-similarity.
2) From Cor. 1 and Prop. 2.2 (the undecidability of $(\approx_\tau)$).
3) Immediately from the definitions.
4) The proof is almost the same as in Prop. 3: the only difference is that we consider not an imitating transition but an imitating parallel step.
5) Note that we can take a union of all resource τp-bisimulations. □

Definition 12. *Let $N = (P, T, F, l)$ be a saturated labelled Petri net with invisible transitions. We say that a relation $B \subseteq \mathcal{M}(P) \times \mathcal{M}(P)$ conforms to the* weak τp-transfer property *if for all $(r, s) \in B$, $t \in T$ s.t. ${}^\bullet t \cap r \neq \emptyset$, there exists an imitating parallel step $U \in \mathcal{M}(T)$ s.t. $l(t) =_\tau l(U)$ and, denoting $M_1 = {}^\bullet t \cup r$ and $M_2 = {}^\bullet t - r + s$, we have $M_1 \xrightarrow{t} M_1'$ and $M_2 \xrightarrow{U} M_2'$ with $(M_1', M_2') \in B^{AT}$.*

In saturated nets the weak τp-transfer property is a necessary and sufficient condition for its extended version, which guarantees the imitation of a parallel step rather than a single transition:

Definition 13. *Let $N = (P, T, F, l)$ be a saturated labelled Petri net with invisible transitions. We say that a relation $B \subseteq \mathcal{M}(P) \times \mathcal{M}(P)$ conforms to the* extended weak τp-transfer property *if for all $(r, s) \in B$ and any parallel step $V \in \mathcal{M}(T)$ s.t. ${}^\bullet V \cap r \neq \emptyset$, there exists an imitating parallel step $U \in \mathcal{M}(T)$ s.t. $l(V) =_\tau l(U)$ and, denoting $M_1 = {}^\bullet V \cup r$ and $M_2 = {}^\bullet V - r + s$, we have $M_1 \xrightarrow{V} M_1'$ and $M_2 \xrightarrow{U} M_2'$ with $(M_1', M_2') \in B^{AT}$.*

Lemma 2. *Let $N = (P, T, F, l)$ be a saturated labelled Petri net with invisible transitions. The relation $B \subseteq \mathcal{M}(P) \times \mathcal{M}(P)$ conforms to the weak τp-transfer property iff it conforms to the extended weak τp-transfer property.*

Proof. ($\Leftarrow$) Since the weak transfer property is a special case of the extended weak transfer property.

($\Rightarrow$) We need to show that for any $(M_1, M_2) \in B^{AT}$ and a parallel step $V = \{t_1, \dots, t_k\} \in \mathcal{M}(T)$ with $M_1 \xrightarrow{V} M_1'$ there exists an imitating parallel step $U \in \mathcal{M}(T)$ with the same visible label $l(V) =_\tau l(U)$ s.t. and $M_2 \xrightarrow{U} M_2'$ and $(M_1', M_2') \in B^{AT}$.

Consider the transition firing $M_1 \xrightarrow{t_1} M_1^1$. From the weak τp-transfer property it follows that this transition has an imitating parallel step $M_2 \xrightarrow{W_1} M_2^1$ such that $(M_1^1, M_2^1) \in B^{AT}$.

Note that $V = \{t_1, \dots, t_k\}$ is a parallel step at marking M_1, hence after the firing of one of these transitions all other are still enabled. Therefore we can repeat the previous reasoning for the new pair of markings $(M_1^1, M_2^1) \in B^{AT}$ and transition t_2. And continue this until t_k:

$$\begin{array}{ccc} M_1 & B^{AT} & M_2 \\ t_1 \downarrow & & \downarrow W_1 \\ M_1^1 & B^{AT} & M_2^1 \\ t_2 \downarrow & & \downarrow W_2 \\ \dots & & \dots \\ t_k \downarrow & & \downarrow W_k \\ M_1' = M_1^k & B^{AT} & M_2^k = M_2' \end{array}$$

At the end we got a sequence of parallel steps

$$M_2 \xrightarrow{W_1} M_2^1 \xrightarrow{W_2} M_2^2 \xrightarrow{W_3} \dots \xrightarrow{W_k} M_2^k = M_2',$$

imitating the firing of parallel step $M_1 \xrightarrow{V} M_1'$. The net is saturated so for any sequence of transitions (note that a parallel step also can be considered as a sequence of transitions) there exists an imitating parallel step U with the same label, precondition and postcondition ($M_2 \xrightarrow{U} M_2'$). □

Note that, unlike the weak transfer property, the extended weak transfer property can not be effectively checked by the search of resource pairs, since the set of parallel steps is infinite.

Theorem 5. *Let $N = (P, T, F, l)$ be a saturated labelled Petri net with invisible transitions. An equivalence relation $B \subseteq \mathcal{M}(P) \times \mathcal{M}(P)$ conforms to the weak τp-transfer property iff B is a resource τp-bisimulation.*

Proof. ($\Leftarrow$) Since the weak τp-transfer property is the τp-transfer property for a bounded (finite) subset of pairs of resources.

($\Rightarrow$) The proof is similar to the proof of Th. 4, with the additional use of Lm. 2. We need to show that B^{AT} conform to the τp-transfer property, i.e. for any $(M_1, M_2) \in B^{AT}$ and $t \in T$ with $M_1 \xrightarrow{t} M_1'$ there exists an imitating parallel step $U \in \mathcal{M}(T)$ with $l(t) = l(U)$, $M_2 \xrightarrow{U} M_2'$ and $(M_1', M_2') \in B^{AT}$.

Consider a pair of markings $(M_1, M_2) \in B^{AT}$. From Lm. 1 this pair can be obtained by a transitive closure of several pairs from B^A (additive closure of B):

$$(H_1, H_2), (H_2, H_3), \dots, (H_{k-1}, H_k) \in B^A, \text{ where } H_1 = M_1, H_k = M_2.$$

Consider the pair (H_1, H_2).

$$(H_1, H_2) = (r_1 + r_2 + \dots + r_l, s_1 + s_2 + \dots + s_l), \text{ where } (r_i, s_i) \in B$$

$H_1 = {}^\bullet t \cup r_1 + F_1$. Due to the weak transfer property for the pair (r_1, s_1) there exists an imitating parallel step $V \in \mathcal{M}(T)$ s.t. $l(t) = l(V), {}^\bullet t \cup r_1 \xrightarrow{t} G_1$ and ${}^\bullet t - r_1 + s_1 \xrightarrow{V} G_2$, where $(G_1, G_2) \in B^{AT}$.

Since ${}^\bullet t \cup r_1 \subseteq H_1$, we can add the resource $F = H_1 - {}^\bullet t \cup r_1$ to preconditions and postconditions:

$$
\begin{gathered}
{}^\bullet t \cup r_1 + F \xrightarrow{t} G_1 + F \\
{}^\bullet t - r_1 + s_1 + F \xrightarrow{V} G_2 + F
\end{gathered}
$$

From the reflexivity of B and the additive closure of B^{AT} the new pair of markings is also decomposable by $B : (G_1 + F, G_2 + F) \in B^{AT}$.

We obtained a new marking $H_1' = {}^\bullet t - r_1 + s_1 + F = H_1 - r_1 + s_1$. Note that it still contains $r_2 + \dots + r_l$. Therefore, we can apply the same reasoning one more time, replacing resource r_2 by the bisimilar resource s_2, now using Lm. 2 and constructing an imitating parallel step not for a transition but for a parallel step V.

Apply this $l - 1$ times. Using transitive closure of B^{AT}, at the end we obtain a parallel step W that can imitate t at marking H_2.

Now proceed to the next pair (H_2, H_3) and repeat the procedure for the parallel step W. And so on, until the last pair (H_{k-1}, H_k). Finally we obtain a parallel step U that can imitate t at marking $H_k = M_2$. □

Thus, in saturated nets the weak τp-transfer property can be used in the construction of resource τp-bisimulation.

4.3. Underapproximation

As in ordinary Petri nets (without invisible transitions), in the case of saturated (saturable) nets with invisible transitions there is a way of constructing an approximation of the maximal resource τp-bisimulation. If we consider not an infinite set of network resources, but only its finite subset, then it will be possible to check the weak τp-transfer property.

Let $N = (P, T, F, l)$ be a saturated labelled Petri net with invisible transitions, $q \in Nat$ — some parameter. By $\mathcal{M}_q(P)$ we denote the set of all resources, containing not more than q tokens in the net: $\mathcal{M}_q(P) = \{r \in \mathcal{M}(P) \; : \; |r| \le q\}$.

Denote by $B_{\tau p}(N, q)$ the union of all resource τp-bisimulations on $\mathcal{M}_q(P)$. Since the union of two resource τp-bisimulations is always a resource τp-bisimulation (Prop. 5.4) we obtain the largest resource τp-bisimulation on $\mathcal{M}_q(P)$.

Since $\mathcal{M}_q(P)$ is finite, we can use the weak transfer property to compute $B_{\tau p}(N, q)$.

Definition 14. *(Underapproximation of largest resource τp-bisimulation)*

Input: *a saturated labelled Petri net with invisible transitions $N = (P, T, F, l)$, parameter $q \in Nat$.*

Output: *Relation $B_{\tau p}(N, q)$.*

Step 1: *Let $C = \emptyset$ — an empty set of pairs (considered as a binary relation over $\mathcal{M}_q(P)$; it will be used as a set of discovered pairs of non-similar resources).*

Step 2: *Compute $B = (\mathcal{M}_q(P) \times \mathcal{M}_q(P)) \setminus C$. Since $\mathcal{M}_q(P)$ is finite the set of pairs B is also finite.*

Step 3: *Compute B_s — the ground basis of B.*

Step 4: *Check, whether B_s conforms to the weak τp-transfer property: it is sufficient to test all non-reflexive elements of B_s (denote a set of all non-reflexive elements of B_s by B_s^{nr}).*

- *If all pairs conforms to the weak τp-transfer property then stop and return B — the bisimulation.*
- *Otherwise there are $(r, s) \in B_s^{nr}$ and $t \in T$ with ${}^\bullet t \cap r \ne \emptyset$, s.t. the firing $M_1 \xrightarrow{t} M_1{}'$ with $M_1 = {}^\bullet t \cup r$ can not be imitated by a parallel step U with the same label and with precondition $M_2 = {}^\bullet t - r + s$ s.t. $M_2 \xrightarrow{U} M_2{}'$ with $(M_1{}', M_2{}') \in B^{AT}$. Add (r, s) and (s, r) to C and go back to Step 2.*

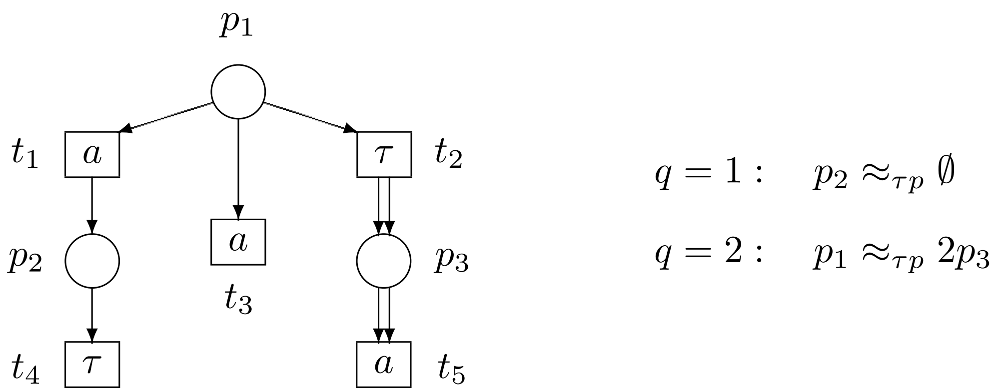

Fig. 6. An example of approximation: resource τp-bisimulation of a saturated Petri net with invisible transitions

Рис. 6. Пример аппроксимации: ресурс τp-бисимуляции насыщенной сети Петри с невидимыми переходами

(termination) For any marking the set of active parallel steps is finite. Also note that the set $\mathcal{M}_q(P) \times \mathcal{M}_q(P)$ is finite. Hence the algorithm always stops.

(correctness) Note that the algorithm stops only if B_s conforms to the weak τp-transfer property. Hence the result is always a resource τp-bisimulation.

(largest equivalence) Assume that not all pairs from the largest resource τp-bisimulation on $\mathcal{M}_q(P)$ are found. Hence each of the lost pairs was removed from the consideration (added to C) at some iteration of algorithm. Consider the first of these iterations. The pair is removed because it doesn't satisfy the weak τp-transfer property w.r.t. the current configuration of B_s. On the other hand, we know that it satisfies the weak τp-transfer property w.r.t. $B_{\tau p}(N, q)$. Since current iteration is first when we remove the "wrong" pair, it is clear that $B_{\tau p}(N, q) \subseteq (B_s)^{AT}$. Hence the pair of resources should satisfy the weak τp-transfer property w.r.t. $(B_s)^{AT}$ – a contradiction.

Denote by $\mathcal{R} = |\mathcal{M}_q(P)|$ the size of the set of considered resources.

At the Step 2 we search through the set of all parallel steps with at most one visible label, that can fire at marking M_2. Each invisible transition can participate in the parallel step at most $|M_2|$ times, since it uses at least one input token.[2] There is also at most one visible transition. Hence we have to check at most $|T||M_2|^{|T|}$ multisets of transitions.

The size of marking $M_2 = {}^{\bullet}t - r + s$ can be evaluated as $O(|s|) = O(q)$.

Using our previous estimations of complexity for ground basis calculation (polynomial w.r.t. $\mathcal{R}$) and the complexity of other steps of algorithm (polynomial w.r.t. the size of the net), we obtain the overall complexity of

$$O(\max\{|P|\,\mathcal{R}^9, |T|^2 q^{|T|} |P|\,\mathcal{R}^7\}).$$

Here the first and the second components of max are estimations for Step 3 and Step 4 respectively. So in the case of nets with invisible transitions the complexity of the algorithm increased significantly (the linear dependence on $|T|$ was replaced by an exponential one). Such a jump is explained by the transition from sets of transitions to multisets.

[2] Without loss of generality we can assume that a net contains no invisible transitions with empty preconditions. In any reachable marking an unobservable sequence of such generating transitions can increase the marking of any of their post-place to a value, exceeding any given natural number. Therefore the places that participate in the postconditions of such generating transitions actually do not affect the observable behavior of the net (and hence the bisimulations), and can be removed along with the corresponding generating transitions.

Consider an example of calculations (Fig. 6). Two subsequent steps are presented: $q = 1$ and $q = 2$. With $q = 1$ we found that resource p_2 is τp-similar to an empty resource (i.e. the place p_2 is redundant). Increasing the parameter ($q = 2$), we obtained one more pair of similar resources $p_1 \approx_{\tau p} 2p_3$.

Proposition 6. *Let $N = (P, T, F, l)$ be a saturated labelled Petri net with invisible transitions. Then:*
1. $\forall q \in Nat \quad (B_{\tau p}(N, q))^{AT} \subseteq (B_{\tau p}(N, q + 1))^{AT}$;
2. $\exists q_f \in Nat : \quad \forall k \in Nat \quad B_{\tau p}(N, q_f + k) = B_{\tau p}(N)$.

Proof. (1) By construction of $B_{\tau p}(N, q)$ for any q the relation $(B_{\tau p}(N, q))^{AT}$ is a largest resource bisimulation s.t. the size of its generating elements (of ground basis) is not greater than q. The union of two resource bisimulations is also a resource bisimulation, hence $B' = (B_{\tau p}(N, q) \cup B_{\tau p}(N, q + 1))^{AT}$ is a resource bisimulation. From the definition of ground basis the generating elements of B' have the size not greater than $q + 1$, therefore $B' = (B_{\tau p}(N, q + 1))^{AT}$.

(2) Since any resource bisimulation is an AT-closed equivalence and therefore it has a finite ground basis (Th. 1). The value of q_f is the size of the largest element of the $B_{\tau p}(N)$ ground basis. □

So at some point q_f the sequence $\{B_{\tau p}(N, q)\}_q$ stabilizes. The problem of q_f computability is still open. The hypothesis is that τp-bisimulation of resources is undecidable and hence q_f is uncomputable.

5. On the approximation of τ-similarity in general nets

If a net is not saturable (see definition in Section 4.2), then the above procedure cannot be applied. However, some straightforward approximations still can be computed.

Consider a parameterized version of the weak τ-transfer property (Def. 6):

Definition 15. *Let $m, n \in Nat \cup \{\infty\}$. We say that a relation $B \subseteq \mathcal{M}(P) \times \mathcal{M}(P)$ conforms to the (m, n)*-weak τ-transfer property *if for all $(r, s) \in B^{AT}$, $t \in T$ s.t. ${}^{\bullet}t \cap r \neq \emptyset$ and $max\{|r|, |s|\} \leq m$, there exists an imitating sequence of transitions $\sigma \in T^*$ s.t. $l(t) =_\tau l(\sigma), |\sigma| \leq n$ and, denoting $M_1 = {}^{\bullet}t \cup r$ and $M_2 = {}^{\bullet}t - r + s$, we have $M_1 \xrightarrow{t} M_1{}'$ and $M_2 \xrightarrow{\sigma} M_2{}'$ with $(M_1', M_2') \in B^{AT}$.*

The first difference is that we check not only elements of B (the base elements of B^{AT}), but all elements of B^{AT} with at most m tokens. The second key property is that we simulate the transition firing not by an arbitrary sequence, but by a sequence with at most n transitions.

Definition 16. *A relation $B \subseteq \mathcal{M}(P) \times \mathcal{M}(P)$ is called an (m, n)*-equivalence *if both B and B^{-1} conform to the (m, n)-weak τ-transfer property.*

Definition 17. *Let N be a net with invisible transitions. Denote by $B_\tau^{(m,n)}(N)$ its largest (m, n)-equivalence.*

Proposition 7. *1. $B_\tau^{(0,0)}(N) = \mathcal{M}(P) \times \mathcal{M}(P)$.*
2. $B_\tau^{(\infty,\infty)}(N) = B_\tau(N)$.

Proof. (1) From the definition of (m, n)-weak τ-transfer property.

(2) Note that in this case $(B_\tau^{(\infty,\infty)}(N))^{AT}$ conforms to the τ-transfer property, hence it is a marking τ-bisimulation. Moreover, it is the largest bisimulation since any union of marking bisimulations is a marking bisimulation. □

Obviously, the limit of sequence $\{B_\tau^{(m,n)}(N)\}_{m,n}$ for $m, n \longrightarrow \infty$ is $B_\tau(N)$. Consider two examples of such a sequence:

Example 1. *For the net depicted in Fig. 2 we have:*

$$B_\tau^{(1,1)}(N) = Id(P)$$

$$B_\tau^{(1,2)}(N) = Id(P) \cup \{(p_1, p_2), (p_2, p_1)\}$$

$$B_\tau^{(2,2)}(N) = Id(P) \cup \{(p_1, p_2), (p_2, p_1)\} \cup \{(p_i, p_j + p_k), (p_j + p_k, p_i) \mid i, j, k \in \{1, 2\}\}$$

$$\dots$$

$$B_\tau^{(m,n)}(N) = B_\tau^{(2,2)}(N)$$

$$\dots$$

$$B_\tau^{(\infty,\infty)}(N) = B_\tau^{(2,2)}(N)$$

Indeed, only the sequences of length 2 can find the similarity between p_1 and p_2. Hence (p_1, p_2) is added only on the second step. On the third step we find out that any non-empty multiset of places is equal to any other non-empty multiset of places — this can be defined by pairs $(p_i, p_j + p_k)$ and $(p_j + p_k, p_i)$ (all other elements can be obtained from these pairs and reflexive pairs with the help of an AT-closure). At the third step the sequence of sets stabilizes.

So as a result we have a non-contracting sequence:

$$(B_\tau^{(1,1)})^{AT} \subset (B_\tau^{(1,2)})^{AT} \subset (B_\tau^{(2,2)})^{AT} = \dots = (B_\tau^{(\infty,\infty)})^{AT}.$$

Example 2. *Consider the net depicted in Fig. 3. Here we have*

$$B_\tau^{(1,2)}(N) = Id(P) \cup \{(p_1, p_2), (p_2, p_1), (p_3, p_4), (p_4, p_3), (p_5, \emptyset), (\emptyset, p_5)\};$$

$$B_\tau^{(2,2)}(N) = Id(P) \cup \{(p_1 + p_4, p_2 + p_3), (p_2 + p_3, p_1 + p_4), (p_5, \emptyset), (\emptyset, p_5)\}.$$

Only at the second step the $(2, 2)$-weak τ-transfer property allowed us to discover the actual non-bisimilarity of resources p_1 and p_2.

The set of pairs is contracting in this particular case:

$$(B_\tau^{(1,2)})^{AT} \supset (B_\tau^{(2,2)})^{AT}.$$

Example 3. *Now consider a net, having two subnets – Fig. 2 and Fig. 3. Obviously, in this case*

$$(B_\tau^{(1,2)})^{AT} \underset{\not\supseteq}{\not\subseteq} (B_\tau^{(2,2)})^{AT}.$$

So, in general the sequence $\{B_\tau^{(m,n)}(N)\}_{m,n} \xrightarrow{m,n\longrightarrow\infty} B_\tau(N)$ is not monotonous even locally. Also note that $B_\tau^{(m,n)}(N)$ can be a subset of $B_\tau(N)$ (Example 1), a superset of $B_\tau(N)$ ($B_\tau^{(1,2)}(N)$ in Example 2) and incomparable to $B_\tau(N)$ (Example 3).

There are two open questions on the structure of $\{B_\tau^{(m,n)}(N)\}_{m,n}$ sequence:

1. Does it always stabilizes at some (m, n)?
2. If not, does it always become monotonous at some point (w.r.t. $m + n$)?

The hypothesis is that the answers are: (1) — negative, (2) — positive. The rationale for this is that $B_\tau^{(m,n)}$ is not always a bisimulation (in contrast to $B_{\tau p}(N, q)$ from the previous section) and hence the infinite "tail" of $\{B_\tau^{(m,n)}(N)\}_{m,n}$ can consist of an infinite sequence of contracting $B_\tau(N)$ overapproximations.

However, as it was shown in the previous examples, the (m, n)-equivalences can still be used in practice as non-trivial approximations of $B_\tau(N)$. The (m, n)-weak τ-transfer property can be effectively checked for any finitely-based candidate B (for example, defined by a ground base) and finite m and n.

Definition 18. *(Computation of an* (m,n)*-equivalence)*

Input: *a labelled Petri net with invisible transitions* $N = (P, T, F, l)$*, parameters* $m, n \in Nat$*.*

Output: *Relation* $B_\tau^{(m,n)}(N)$*.*

Step 1: *Compute a tree* Tr *of all possible ground bases (except the trivial reflexive basis* $Id(P)$*) having the size of their elements not greater than* m*. In this tree a basis* B_s *is a parent node for a basis* B_s' *iff* $(B_s')^{AT} \subset (B_s)^{AT}$*.*

Step 2: *Using breadth-first search, take the next node* B_s *from* Tr *and check, whether* B_s *conforms to the* (m,n)*-weak* τ*-transfer property.*

- *If all pairs conforms to the* (m,n)*-weak* τ*-transfer property then stop and return* B_s*.*
- *Otherwise there are* $(r,s) \in (B_s)^{AT}$ *with* $max\{|r|,|s|\} \leq m$ *and* $t \in T$ *with* ${}^\bullet t \cap r \neq \emptyset$*, such that the firing* $M_1 \xrightarrow{t} M_1{}'$ *with* $M_1 = {}^\bullet t \cup r$ *can not be imitated by a sequence* $\sigma \in T^*$ *of (at most) n transitions with label* $l(t)$ *and precondition* $M_2 = {}^\bullet t - r + s$ *such that* $M_2 \xrightarrow{\sigma} M_2{}'$ *with* $(M_1{}', M_2{}') \in B^{AT}$*. In this case go back to the Step 2.*

Step 3: *Return* $Id(P)$*.*

(termination) The resource size is bounded by m, the length of firing sequences is bounded by n, the (m,n)-weak τ-transfer property can be checked in a finite number of steps. The tree Tr is also finite. Hence the algorithm always stops.

(correctness) The construction of the tree Tr implies that the largest (m,n)-equivalence is always the closest to the root (note that it contains all other (m,n)-equivalences). Hence the algorithm (breadth-first search) finds it first.

Note that this "algorithm" is simple, but highly ineffective. There are four non-polynomial procedures: Tr computation, Tr search, the resource pair combination and the transition sequence search.

6. Conclusion

The proposed methods for finding pairs of similar resources are of particular interest for certain applications, such as model reduction (shrinking the net without affecting its behavior) and adaptive process management (resource relocation in the aftermath of some acute events). In addition, the use of resource bisimulation allows one to reduce a Petri net with conservation of its behavior. This reduction is important when analyzing properties of the Petri net, since the computational complexity of the majority of algorithms used in analysis depends exponentially on the size of the net.

Important open questions concern decidability and complexity of related algorithmic problems. For example, we have already shown that all types of resource similarity (ordinary, τ-, τp-) are undecidable. On the other hand, the problem of $B(N)$ (and $B_\tau(N)$, and $B_{\tau p}(N)$) computability is still open. We have introduced only the underapproximations.

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