Модел. и анализ информ. систем. Т. **19**, № **6** (2012) 127–136

UDC 514.772.35

Continuous Flattening of a Regular Tetrahedron with Explicit Mappings

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received September 15, 2012

Keywords: Continuous Flattening, Regular Tetrahedra, Polyhedra, Paper Folding

We proved in [10] that each Platonic polyhedron P can be folded into a flat multilayered face of P by a continuous folding process of polyhedra. In this paper, we give explicit formulas of continuous functions for such a continuous flattening process in $\mathbb{R}^3$ for a regular tetrahedron.

The article is published in the author's wording.

1. Introduction

We use the terminology *polyhedron* for a closed polyhedral surface which is permitted to touch itself but not self-intersect (and so a doubly covered polygon is a polyhedron). A *flat folding* of a polyhedron is a folding by creases into a multilayered planar shape ([7], [8]).

A. Cauchy [4] in 1813 proved that any convex polyhedron is rigid: precisely, if two convex polyhedra P, P' are combinatorially equivalent and their corresponding faces are congruent, then P and P' are congruent. By removing the condition of convexity, R. Connelly [5] in 1978 gave an example of a (non-convex) flexible polyhedron: precisely, there is a continuous family of polyhedra $\{P_t : 0 \leq t \leq 1\}$ such that for every $t \neq 0$, the corresponding faces of P_0 and P_t are congruent while polyhedra P_0 and P_t are not congruent. (See also [6].) After then I. Sabitov [15] in 1998 proved that the volume of any polyhedron is invariant under flexing: precisely, if there is a continuous family of polyhedra $\{P_t : 0 \leq t \leq 1\}$ such that, for every t, the corresponding faces of P_0 and P_t are congruent, then the volumes P_0 and P_t are equal for all $0 \leq t \leq 1$. (See also [14].)

A. Milka [12] in 1994 showed that any polyhedron admits a continuous (isometric) deformation by using moving edges, and that all Platonic polyhedra can be changed in

[1]Partially supported by Grand-in-Aid for Scientific Research (No.23540098), JSPS.
[2]Partially supported by Grand-in-Aid for Scientific Research (No.23540160), JSPS.

their exterior shapes. He called such a deformation a *linear bending*. I. Sabitov [16] in 2000 explained on a class of deformations of polyhedra without the face-rigidity condition, and he introduced an example of a regular tetrahedron whose vertex is pushed in continuously with moving creases (which he called "swimming" edges). Also T. Banchoff [2] discussed on a continuous isometric deformation of a square polyhedral torus. D. Bleeker [3] showed that any convex polyhedron in $\mathbb{R}^3$ admits a continuous isometric deformation such that the volume increases. (See also [13].)

In our previous works, we discussed on flattening of convex polyhedra by a continuous (isometric) deformation which we call a *continuous folding process* (see Definition 1). For such a deformation, its faces must be changed by moving creases as its volume decreases. We proved in 2010 that each Platonic polyhedron is fattened by a continuous flat folding process of polyhedra onto its original face ([10]). Furthermore we showed with C. Vîlcu in 2011 that any convex polyhedron admits infinitely many continuous folding processes, by using cut loci and Alexandrov's gluing theorem ([1], [11]).

In this paper, we give explicit formulas of continuous functions for a continuous flat folding process in the case of a regular tetrahedron. We leave such calculation for other Platonic polyhedra in future work.

2. Explicit formulas of continuous flat folding mappings

We proved the following lemma which played a key role for proofs in [10].

Lemma. *For any $l\,(0 \le l \le |AC|)$ and any $m\,(0 \le l \le |BD|)$ of a rhombus $ABCD$ with the center M, there are points Q, R on the line segment AC which satisfy the following:*

(i) $|QM| = |RM|$,

(ii) by folding R with creases $\{AM, BM, DM, RB, RD, MR, RC\}$, $\triangle BMR$ and $\triangle DMR$ are folded onto $\triangle BQM$ and $\triangle DQM$ respectively, and

(iii) $dist\{A', C'\} = l$ and $dist\{B', D'\} = m$, where X' in the folded rhombus is the corresponding point to a point X in the rhombus $ABCD$ and $|YZ|$ means the length of a line segment YZ. (see Fig. 1).

Definition 1. Let P be a polyhedron in the Euclidean space $\mathbb{R}^3$. We say that a family of polyhedra $\{P_t : 0 \le t \le 1\}$ is a *continuous folding process* from $P = P_0$ to P_1 if it satisfies the following conditions:

(1) for each $0 \le t \le 1$, there exists a polyhedron P_t' obtained from P by subdividing some faces of P (i.e., some faces of P_t' may be included in the same face of P, but P_t' is congruent to P) such that P_t is combinatorially equivalent to P_t' and the corresponding faces of P_t' and P_t are congruent,

(2) the mapping $[0, 1] \ni t \longmapsto P_t \in \{P_t : 0 \le t \le 1\}$ is continuous.

Theorem. *The regular tetrahedron in $\mathbb{R}^3$ with vertices $O = (0, 0, 0)$, $A = (2/\sqrt{3}, 0, -2\sqrt{2}/\sqrt{3})$, $B = (\sqrt{3}, 1, 0)$, and $C = (\sqrt{3}, -1, 0)$ is flattened explicitly by a continuous folding process of polyhedra $\{P_t : 0 \le t \le 1\}$ which satisfies the following:*
(i) the line segment OM, where M is the midpoint of the edge BC, is fixed on the x-axis (see Fig. 2(1)(2)),
(ii) two faces $\triangle OAB$ and $\triangle OAC$ have no crease,

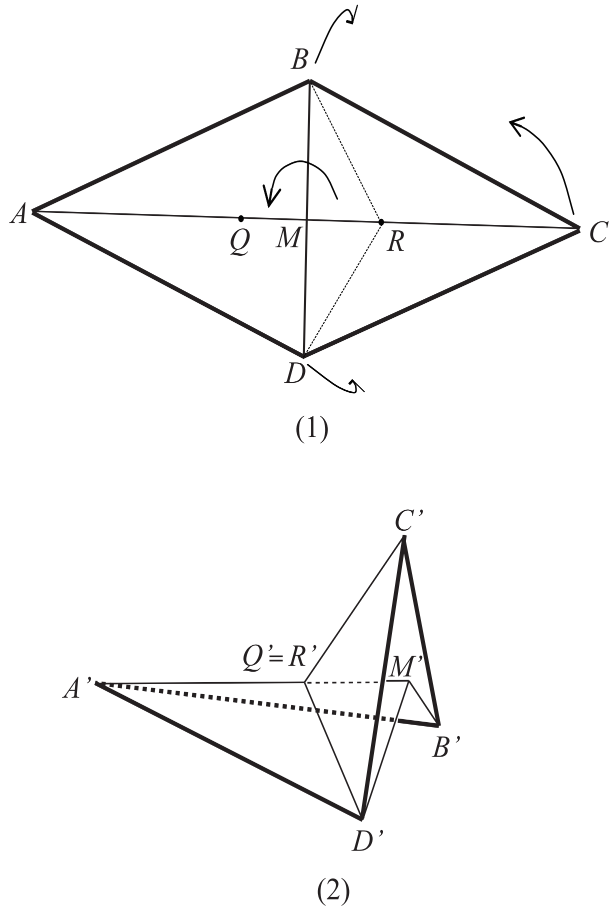

Figure 1. How to fold a rhombus:(1) a rhombus (2) an example of a folded rhombus.

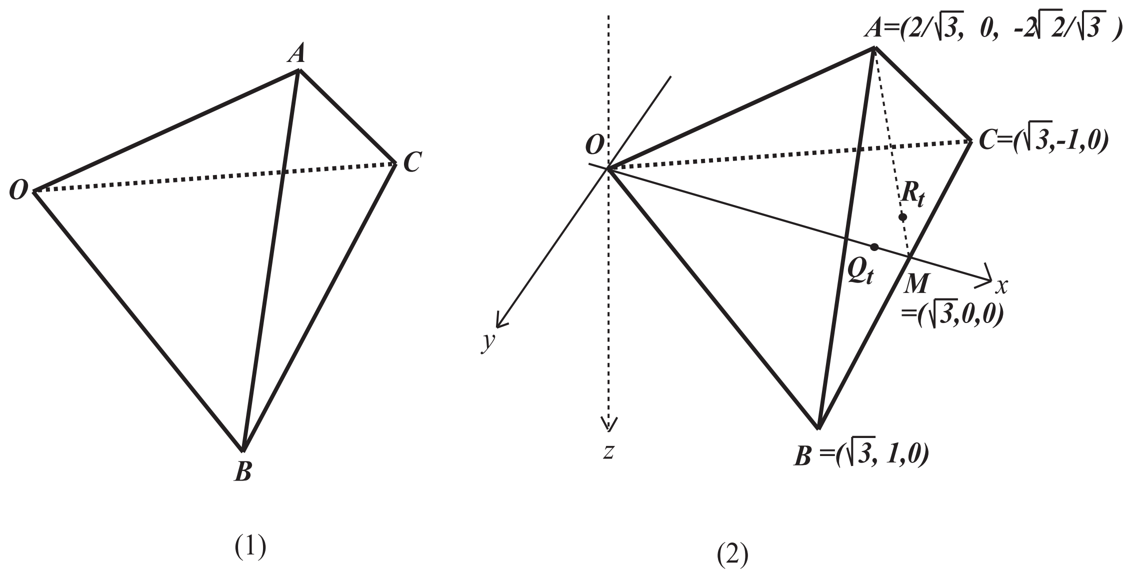

Figure 2. (1) A tetrahedron (2) the tetrahedron with coordinates.

(iii) there are points Q_t $(0 \leq t \leq 1)$ on OM and R_t on AM such that for each t the face $\triangle ABC$ is divided into four triangles $\triangle ABR_t$, $\triangle ACR_t$, $\triangle BR_tM$ and $\triangle CR_tM$, and that $\triangle B_t(R_t)'M$ and $\triangle C_t(R_t)'M$ are attached to $\triangle B_tQ_tM$ and $\triangle B_tQ_tM$ respectively, where we denote by A_t, B_t, C_t and $(R_t)'$ the points on P_t corresponding to points A, B, C and R_t respectively (see Fig. 3), and
(iv) explicit coordinates of A_t, B_t, C_t and Q_t are

$$A_t = (\frac{6 + 2s\sqrt{6 + 3s^2}}{\sqrt{3}(3 + s^2)},\ 0,\ \frac{2(s - \sqrt{6 + 3s^2}}{3 + s^2}),$$

$$B_t = (\sqrt{3},\ \sqrt{1 - s^2},\ s),$$

$$C_t = (\sqrt{3},\ -\sqrt{1 - s^2},\ s),$$

$$Q_t = (\frac{\sqrt{3}(3 + s^2)}{3 + s\sqrt{6 + 3s^2}},\ 0,\ 0)$$

where $s = \sin\frac{\pi}{2}t\,(0 \leq t \leq 1)$ (see Figure 2 and Figure 3), and
(v) P_1 is a flat folded state of P (see Fig. 4).

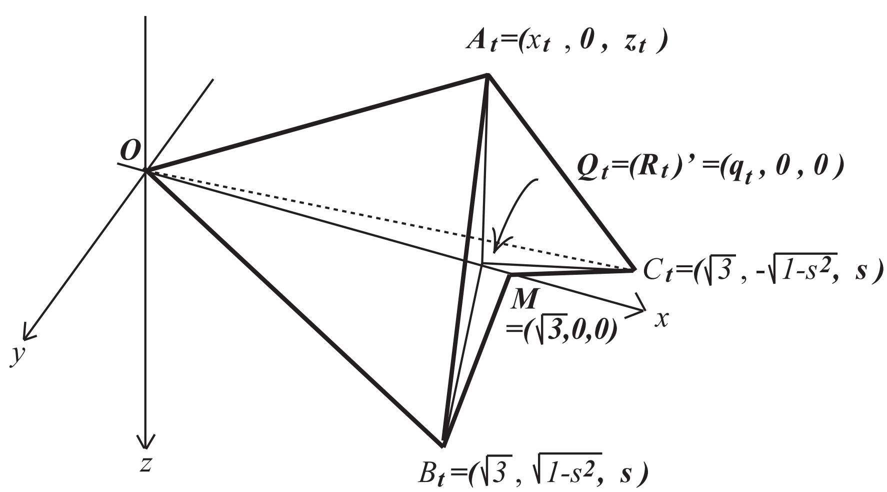

Figure 3. $P_t\,(0 \leq t \leq 1)$ where $s = \sin\frac{\pi}{2}t$.

Proof. Let P be a regular tetrahedron with the edge length two, whose vertices are O, A, B, C in the xyz-space $\mathbb{R}^3$ with coordinates $O = (0, 0, 0)$, $A = (2/\sqrt{3}, 0, -2\sqrt{2}/\sqrt{3})$, $B = (\sqrt{3}, 1, 0)$, and $C = (\sqrt{3}, -1, 0)$. Denote by $M = (\sqrt{3}, 0, 0)$ the midpoint of the edge BC.

Get a crease of the line segment OM on the face $\triangle OBC$, fold $\triangle OBC$ continuously into halves until B and C meets on the xz-plane. According to such folding, we define a

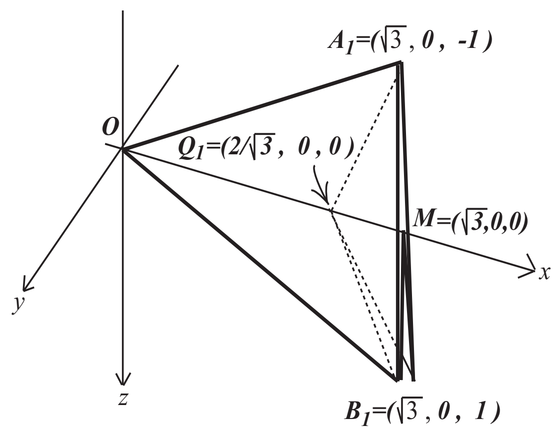

Figure 4. The flat folded state P_1 of the regular tetrahedron P.

continuous folding process of P_t $(1 \leq t \leq t)$ for P which satisfies the following conditions $(i) - (v)$ in Theorem. Fix the line segment OM for all $0 \leq t \leq 1$ and denote $s(t) = \sin \frac{\pi}{2} t \, (0 \leq t \leq 1)$. Rotate $triangleOBM$ and $triangleOCM$ about OM as follows: for each $t \, (0 \leq t \leq 1)$

$$B_t = (\sqrt{3},\ \sqrt{1-s^2},\ s),$$

$$C_t = (\sqrt{3},\ -\sqrt{1-s^2},\ s),$$

where $s = s(t) = \sin \frac{\pi}{2}$ and by $X_t \in P_t$ (for $X = A,\ B,\ C$) and $(R_t)'$ the corresponding point to points X and R_t on P..

Since we get no crease on $\triangle OAB$ and $\triangle OAC$, the distances between O, A_t and A_t, B_t are $dist(O,\ A_t) = 2$ and $dist(A_t,\ B_t) = 2$. Then the coordinates of $A = (x_t,\ 0,\ z_t)$ satisfy

$$(x_t)^2 + (z_t)^2 = 4 \tag{1}$$

$$(\sqrt{3} - x_t)^2 + (1 - s^2) + (s - z_t)^2 = 4 \tag{2}$$

By subtracting (2) from (1) in each side of the equalities, it follows

$$(x_t)^2 + (z_t)^2 - \{(\sqrt{3} - x_t)^2 + (1 - s^2) + (s - z_t)^2\} = 0.$$

Hence,

$$x_t = \frac{2 - s \cdot z_t}{\sqrt{3}}. \tag{3}$$

Substituting the equation (3) to the equation (1), by $z_t \leq 0$

$$z_t = \frac{2(s - \sqrt{6 + 3s^2})}{3 + s^2}, \tag{4}$$

and hence

$$A_t = (\frac{6 + 2s\sqrt{6 + 3s^2}}{\sqrt{3}(3 + s^2)},\ 0,\ \frac{2(s - \sqrt{6 + 3s^2})}{3 + s^2}).$$

By Lemma 1 note that Q_t is the intersection point of the orthogonal bisector of OA_t with the line segment OM. Let $Q_t = (q_t, 0, 0)$. Since the midpoint N_t of OA_t is $N_t = (x_t, 0, z_t)$ and the inner product of the vector OA and the vector N_tQ_t is zero, we have

$$(\frac{x_t}{2} - q_t) \cdot x_t + \frac{(z_t)^2}{2} = 0.$$

By (1) it holds

$$q_t = \frac{2}{x_t}.$$

Therefore we get

$$Q_t = (\frac{\sqrt{3}(3+s^2)}{3+s\sqrt{6+3s^2}}, 0, 0).$$

□

3. Another continuous flattening

In Theorem 1, we pushed the face $\triangle ABC$ *inside* to flatten the regular tetrahedron $OABC$ (see Fig. 3 and Fig. 4). If we push the face $\triangle ABC$ *outside*, can we still flatten the tetrahedron continuously? We show there is a continuous folding process for such a flattening. Denote by G and H the centers of gravity for $\triangle ABC$ and $\triangle OAB$ (see Fig. 5(1)). Then the quadrilateral $HBGC$ is a rhombus which is folded in a halfway with a crease BC (see Fig. 5(2)). By applying Lemma 1, there are moving creases BR_t and CR_t $(0 \leq t \leq 1)$ with points R_t on the line segment MG such that $dist\{B_t, C_t\}$ decreases to zero and $dist\{H, (R_t)'\}$ increases to $2/\sqrt{3}$ simultaneously for given distances (see Fig. 5(3) and Fig. 3(4)) where we use the same notations for A_t, B_t, C_t and $(R_t)'$ as the ones used in the proof of Theorem 1.

Let fold $\triangle OAB$ and $\triangle OAC$ similar way to the one defined in the proof of Theorem 1. Then a continuous folding process of P_t $(0 \leq t \leq 1)$ is obtained as follows:

(i) the coordinates of $G_t \in P_t$ (which is the corresponding point to G) is uniquely determined by the equation (see Fig. 6)

$$|A_tG_t| = |B_tG_t| = |C_tG_t| = 2/\sqrt{3},$$

(ii) by using similar argument to the one used in the proof of Theorem 1, we can calculate the coordinates of $(R_t)' \in P_t$.

Finally the tetrahedron is flattened in a shape showed in Fig. 7.

4. Further research

It is almost obvious that the area of creases for the continuous flattening process discussed in the section 3 is smaller than the one showed in Theorem 1. We asked the following problem in [10]

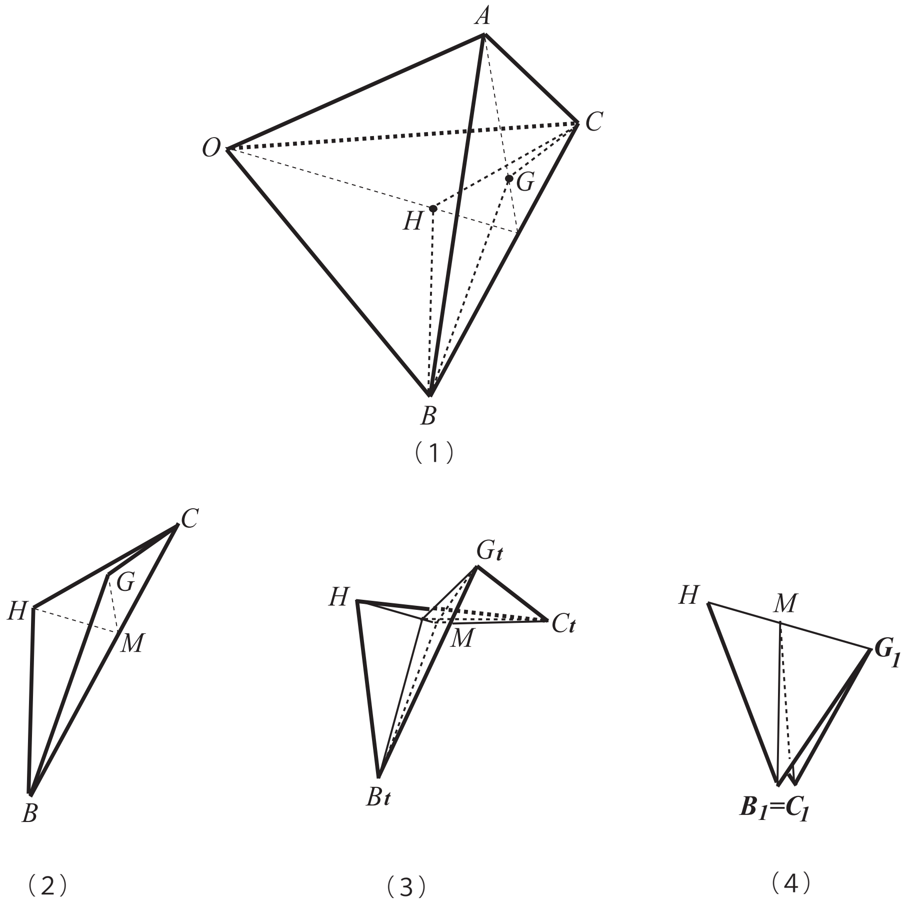

Figure 5. A continuous folding process for a rhombus.

Question. *What is the minimum area of creases which are used for a continuous flattening of each Platonic polyhedron?*

For the case of flattening showed in Theorem 1, the area is $1/\sqrt{3}$, and for the one showed in the section 3, Ko-ichi Hirata [9] got an approximate value $(\sqrt{3}-\sqrt{2})/2$ by using Mathematica.

Acknowledgement. The authors are indebted to I. Sabitov for his useful suggestions and encouragement to prepare this paper, and N. Dolbilin for giving us the opportunity to discuss on this matter in the conference "Discrete geometry" held in Yaroslavl.

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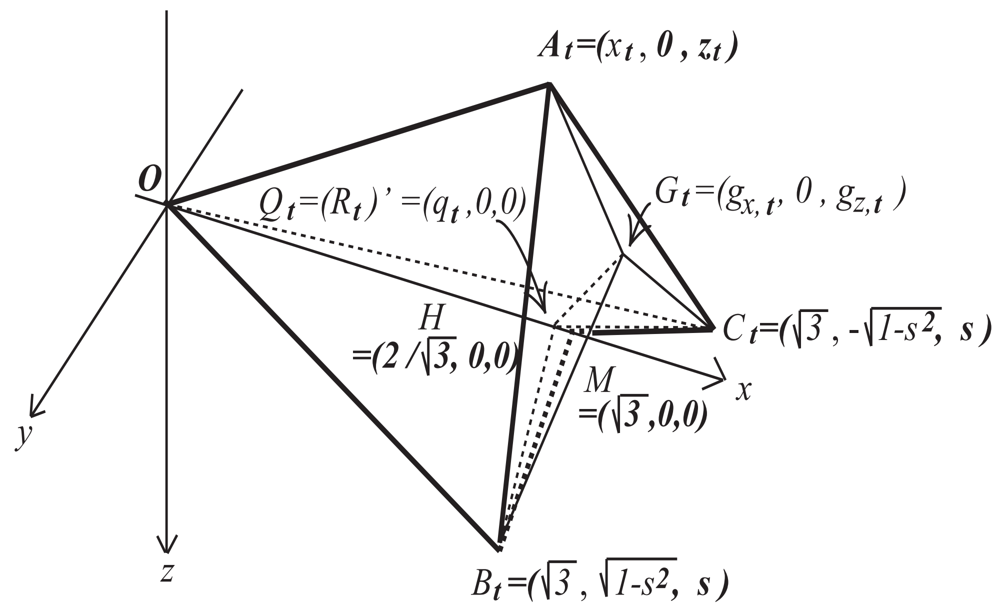

Figure 6. Another continuous folding process of $P_t\,(0 \leq t \leq 1)$ where $s = \sin \frac{\pi}{2} t$.

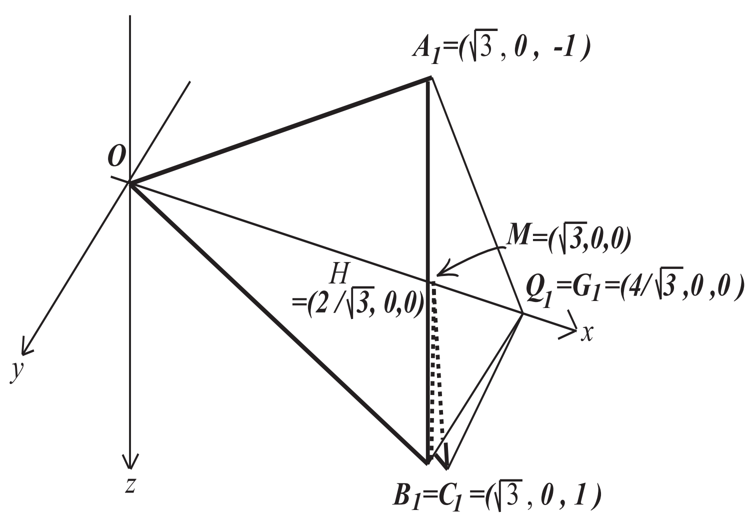

Figure 7. The flat folded state P_1 by another flattening.

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Непрерывное уплощение правильного тетраэдра точными отображениями

Джин-ичи Ито, Чи Нара

Ключевые слова: непрерывное складывание, правильные тетраэдры, многогранники, складывание бумаги

В статье [10] нами доказано, что любой правильный многогранник P допускает непрерывное (изометричное) складывание (или разглаживание) на плоскость. В настоящей статье мы приводим явные формулы непрерывных функций такого процесса складывания для правильного тетраэдра в $\mathbb{R}^3$. Статья публикуется в авторской редакции.

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