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A Uniform Asymptotical Upper Bound for the Variance of a Random Polytope in a Simple Polytope

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The present paper contains a sketch of the proof of an upper bound for the variance of the number of hyperfaces of a random polytope when the mother body is a simple polytope. Thus we verify a weaker version of the result in [1] stated without a proof. The article is published in the author's wording.

Let X be a convex body in $\mathbb{R}^d$. A *random polytope* $P_n(X)$ is a convex hull of n independent uniformly distributed points in X.

Starting with the paper by Rényi and Sulanke [3], random polytopes have been a popular object for research in stochastic geometry. There are numerous results concerning different functionals of $P_n(X)$, and the most important of these functionals are the volume $\operatorname{vol} P_n(X)$ and the components of the f-vector $f_i(P_n(X))$.

Since 1990s a lot of research has been done on the distributional properties of stochastic variables of type $A(P_n(X))$, where A is a given functional of a polytope. Most of this research uses different estimates for $\operatorname{Var} A(P_n(X))$. In [2] Wieacker and Weil state that the determination of the variance is a major open problem.

Let X be a polytope. The paper [4] estimates $\operatorname{Var} f_{d-1}(P_n(X))$ as follows:

$$\mathsf{E}\, f_{d-1}(P_n(X)) \ll \operatorname{Var} f_{d-1}(P_n(X)) \ll C(X)\, \mathsf{E}\, f_{d-1}(P_n(X)),$$

where $C(X)$ depends on the combinatorial structure of X. However, it seems to be natural to change the bounds so that their ratio would be independent of X.

We obtain an inequality with coincident upper and lower bounds in the case of a *simple* polytope X. Namely, we prove the following theorem

Theorem 1. *There exist positive real numbers C_1, C_2 such that for every simple polytope X and every positive integer $n > n_0(X)$ one has*

$$C_1\, \mathsf{E}\, f_{d-1}(P_n(X)) < \operatorname{Var} f_{d-1}(P_n(X)) < C_2\, \mathsf{E}\, f_{d-1}(P_n(X)),$$

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Consider a *Poisson polytope* $\Pi_\lambda(X)$, i.e. the convex hull of a Poisson process in X of intensity λ. If we prove the inequality

$$C_1' \,\mathsf{E}\, f_{d-1}(\Pi_\lambda(X)) < \operatorname{Var} f_{d-1}(\Pi_\lambda(X)) < C_2' \,\mathsf{E}\, f_{d-1}(\Pi_\lambda(X)) \tag{1}$$

for $\lambda > \lambda_0(X)$, then theorem 1 will immediately follow from the result [5] by J. Pardon. So, our goal is to prove (1).

For a generic d-tuple $\mathbf{x} = (x_1, x_2, \dots, x_d) \in X^d$ the hyperplane $\operatorname{aff}(x_1, x_2, \dots, x_d)$ is uniquely defined and splits X into 2 *caps*. Denote by $\mathcal{C}(\mathbf{x})$ the smaller of those caps. Define

$$c(\mathbf{x}) = \operatorname{vol} \mathcal{C}(\mathbf{x}) \quad \text{and} \quad i(\mathbf{x}, \mathbf{y}) = \operatorname{vol}(\mathcal{C}(\mathbf{x}) \cap \mathcal{C}(\mathbf{y})).$$

Let $S \subset X^d$ be a measurable set invariant under all permutations of the d-tuple. Say that a facet of $\Pi_\lambda(X)$ *corresponds* to S if the d-tuple of its vertices belongs to S and the smaller cap defined by its affine hull contains no vertex of $\Pi_\lambda(X)$. Denote by $f_{d-1}(S)$ the number of facets of $\Pi_\lambda(X)$ corresponding to S. Let $a_i(S)$ be the number of ordered pairs (F_1, F_2) such that F_1, F_2 are facets of $\Pi_\lambda(X)$ corresponding to S with exactly i common vertices.

Since for every stochastic variable ξ there is an identity

$$\operatorname{Var} \xi = \mathsf{E}\, \xi^2 - (\mathsf{E}\, \xi)^2,$$

we immediately get

$$\operatorname{Var} f_{d-1}(S) = \mathsf{E}\, a_1(S) + \mathsf{E}\, a_2(S) + \ldots + \mathsf{E}\, a_d(S) + (\mathsf{E}\, a_0(S) - (\mathsf{E}\, f_{d-1}(S))^2). \tag{2}$$

Let $S_i \subset X^{2d-i}$ be the set of $(2d-i)$-tuples $\mathbf{z} = (z_1, z_2, \dots, z_{2d-i})$ such that

1. $\mathbf{x}(\mathbf{z}) = (z_1, z_2, \dots, z_d)$ and $\mathbf{y}(\mathbf{z}) = (z_{d-i+1}, z_{d-i+2}, \dots, z_{2d-i})$ belong to S.
2. No point z_j belongs to $\mathcal{C}(\mathbf{x}(\mathbf{z})) \cup \mathcal{C}(\mathbf{y}(\mathbf{z}))$.

According to Slivnyak-Mecke formula (see, for example [6]),

$$\mathsf{E}\, a_i(S) = \frac{\lambda^{2d-i}}{i!(d-i)!^2} \int_{S_i} \cdots \int e^{-\lambda\{c(\mathbf{x}(\mathbf{z})) + c(\mathbf{y}(\mathbf{z})) - i(\mathbf{x}(\mathbf{z}), \mathbf{y}(\mathbf{z}))\}} \, d\mathbf{z}. \tag{3}$$

By the same formula,

$$\mathsf{E}\, f_{d-1}(S) = \frac{\lambda^d}{d!} \int_S \cdots \int e^{-\lambda c(\mathbf{x})} \, d\mathbf{x}. \tag{4}$$

Applying to (4) the identity $\left(\int_S \cdots \int G(\mathbf{x}) \, d\mathbf{x} \right)^2 = \int_{S \times S} \cdots \int G(\mathbf{x}) G(\mathbf{y}) \, d\mathbf{x} d\mathbf{y}$, we get

$$(\mathsf{E}\, f_{d-1}(S))^2 = \frac{\lambda^{2d}}{d!^2} \int_{S \times S} \cdots \int e^{-\lambda\{c(\mathbf{x}) + c(\mathbf{y})\}} \, d\mathbf{x} d\mathbf{y}. \tag{5}$$

Now we can rewrite the last term of (2) as follows:

$$\mathsf{E}\, a_0(S) - (\mathsf{E}\, f_{d-1}(S))^2 = \frac{\lambda^{2d}}{d!^2} \int\limits_{S_0}\cdots\int e^{-\lambda\{c(\mathbf{x})+c(\mathbf{y})-i(\mathbf{x},\mathbf{y})\}} \left(1 - e^{-\lambda i(\mathbf{x},\mathbf{y})}\right) d\mathbf{x}d\mathbf{y} - \frac{\lambda^{2d}}{d!^2} \int\limits_{S\times S\setminus S_0}\cdots\int e^{-\lambda\{c(\mathbf{x})+c(\mathbf{y})\}} d\mathbf{x}d\mathbf{y}. \quad (6)$$

Finally, we have obtained the integral expressions for all the terms of (2).

Suppose $\{S_1, S_2, \ldots, S_N\}$ is a partition of X^d. Then we can observe that

$$f_{d-1}(\Pi_\lambda(X)) = f_{d-1}(S_1) + f_{d-1}(S_2) + \ldots + f_{d-1}(S_N) + \hat{f}_{d-1}, \quad (7)$$

where $\hat{f}_{d-1}$ is the number of facets of $\Pi_\lambda(X)$ such that a greater cap of this facet contains no vertex of $\Pi_\lambda(X)$. Our goal now is to construct a nice partition of X^d.

Recall that X is a simple polytope. Enumerate all the faces of X: $F_1, F_2, \ldots F_N$ so that $i > j$ whenever $\dim F_i > \dim F_j$. (We treat X as its own face, so $F_N = X$.)

For every generic d-tuple $\mathbf{x} \in X^d$ find the face F_i with the 2 properties:

1. $\dim \operatorname{aff}(\operatorname{vert} F_i \cap \mathcal{C}(\mathbf{x})) = \dim F_i$.
2. i is the maximal number to satisfy condition 1.

Let S_i be the set of all d-tuples corresponding to F_i. Write down all the estimates we need to proceed.

$$\begin{aligned}
\mathsf{E}\, f_{d-1}(S_i) &\lesseqgtr \ln^d \lambda, && \text{if } F_i \text{ is a vertex;}\\
\mathsf{E}\, f_{d-1}(S_i) &< C(X)\cdot ln^{d-1}\lambda, && \text{if } F_i \text{ is not a vertex;}\\
\mathsf{E}\, \hat{f}_{d-1} &< C(X)o(1);\\
\operatorname{Var} f_{d-1}(S_i) &\ll \ln^d \lambda, && \text{if } F_i \text{ is a vertex;}\\
\operatorname{Var} f_{d-1}(S_i) &< C(X)\cdot ln^{d-1}\lambda, && \text{if } F_i \text{ is not a vertex;}\\
\operatorname{Var} \hat{f}_{d-1} &< C(X)o(1);\\
\operatorname{Cov}(f_{d-1}(S_i), f_{d-1}(S_j)) &= 0, && \text{if } F_i \text{ and } F_j \text{ are vertices with no adjoining edge.}
\end{aligned}$$

For all other covariations apply the inequality $\operatorname{Cov}(\xi_1, \xi_2) \le \sqrt{\operatorname{Var}\xi_1 \operatorname{Var}\xi_2}$.

Writing down $\operatorname{Var} f_{d-1}(\Pi_\lambda(X))$ as the variance of the sum (7), we see that

$$\operatorname{Var} f_{d-1}(\Pi_\lambda(X)) \ll M \ln^d \lambda \lesseqgtr \mathsf{E}\, f_{d-1}(\Pi_\lambda(X)),$$

where M is the number of vertices of X. This gives the upper bound for the variance. The lower bound is due to Bárány and Reitzner (see [4]).

Remark. Probably, the analogue of Theorem 1 is true for all convex polytopes X (and this is exactly the statement left in [1] without a proof). However, the author was not able to get a satisfactory uniform estimate for $\operatorname{Var} f_{d-1}(S_i)$ when F_i is a vertex. We can go even further and conjecture the analogue of Theorem 1 for all convex bodies. However, the methods described above are not applicable in the most general case.

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Равномерная асимптотика верхней границы дисперсии для случайного многогранника

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Ключевые слова: случайный многогранник, f-вектор, дисперсия

Содержится развернутый план доказательства равномерной оценки дисперсии числа гиперграней случайного многогранника в случае, если объемлющее тело — простой многогранник. Таким образом, доказана ослабленная версия результата, оставленного в [1] без доказательства. Статья публикуется в авторской редакции.

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