Модел. и анализ информ. систем. Т. **20**, № **6** (2013) 129–134

UDC 511.6

A Definition of Type Domain of a Parallelotope

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received October 10, 2013

Keywords: parallelotope, type domain, zonotope

Each convex polytope $P = P(\alpha)$ can be described by a set of linear inequalities determined by vectors p and right hand sides $\alpha(p)$. For a fixed set of vectors p, a type domain $\mathcal{D}(P_0)$ of a polytope P_0 and, in particular, of a parallelotope P_0 is defined as a set of parameters $\alpha(p)$ such that polytopes $P(\alpha)$ have the same combinatorial type as P_0 for all $\alpha \in \mathcal{D}(P_0)$.

In the second part of the paper, a facet description of zonotopes and zonotopal parallelotopes are given.

The article is published in the author's wording.

1. Properties of type domains

A *type* of a polytope P, in particular, of a parallelotope, is an isomorphism class of the partial ordered set of all faces of P.

There corresponds a Voronoi polytope $P_V(f)$ to each positive semidefinite quadratic form f. Voronoi defined in his famous paper [1] an L-type domain of a Voronoi polytope $P_V(f_0)$ as a set of quadratic forms f such that polytopes $P_V(f)$ have the same type as $P_V(f_0)$. Voronoi conjectured in [1] that each parallelotope is affinely equivalent to a Voronoi polytope. Since the Voronoi conjecture is not yet proved, it is useful to define a type domain of a parallelotope not using quadratic forms.

Call a face of codimension 1 by a *facet*. Each n-dimensional convex centrally symmetric polytope P can be described by the following system of inequalities

$$P = P(\alpha) = \{x \in \mathbb{R}^n : \langle p, x\rangle \le \alpha(p),\ p \in \mathcal{P}\}, \tag{1}$$

where $\mathcal{P} \subset \mathcal{R}^n$ is a set of vectors including all facet vectors such that if $p \in \mathcal{P}$, then $-p \in \mathcal{P}$. Here $\langle p, x\rangle$ is scalar product of vectors $p, x \in \mathbb{R}^n$. The function $\alpha \in \mathbb{R}^{\mathcal{P}}_+$ is symmetric and non-negative, i.e. $\alpha(-p) = \alpha(p) \ge 0$ for all $p \in \mathcal{P}$. Call the function α by *parameter.* Let $\mathcal{P}(P) \subseteq \mathcal{P}$ be a set of all facet vectors of P. Suppose that, for any $p \in \mathcal{P}$, the following hyperplane

$$H(\alpha, p) = \{x \in \mathbb{R}^n : \langle p, x\rangle = \alpha(p)\} \tag{2}$$

supports P at a face G, i.e. $H(\alpha, p) \cap P = G$. The face G is a facet if $p \in \mathcal{P}(P)$.

Actually, I consider a family of polytopes $P(\alpha)$ for distinct α, but for a fixed set of vectors $\mathcal{P}$. Each of polytopes of the family is obtained from any other by parallel shifts of supporting hyperplanes $H(\alpha, p)$.

A *type domain* $\mathcal{D}(P)$ of a polytope P is a set of all parameters α such that polytopes $P(\alpha)$ have the same type as P for all $\alpha \in \mathcal{D}(P)$. The domain $\mathcal{D}(P)$ is determined by equalities and inequalities between values $\alpha(p)$ for distinct p.

Let $P = P(\alpha)$ be a polytope described by (1). Let $\mathcal{P}(G) \subseteq \mathcal{P}(P)$ be a set of facet vectors of all facet containing G. The following assertion describes some equalities between parameters $\alpha(p)$.

Proposition 1. *Let G be a k-dimensional face of a polytope P. Then*

(i) if $t \in \mathcal{P} - \mathcal{P}(G)$ is such that the hyperplane $H(\alpha, t)$ supports P at the face G, then

$$\alpha(t) = \sum_{p \in \mathcal{P}(G)} \mu_t(p)\alpha(p),$$

where $\mu_t(p) \geq 0$ are coefficients of the decomposition $t = \sum_{p \in \mathcal{P}(G)} \mu_t(p)p$ of the vector t by the facet vectors $p \in \mathcal{P}(G)$;

(ii) if $|\mathcal{P}(G)| > n - k$, then

$$\sum_{p \in \mathcal{P}(G)} \mu(p)\alpha(p) = 0,$$

where $\mu(p)$ are coefficients of a linear dependence $\sum_{p \in \mathcal{P}(G)} \mu(p)p = 0$ between vectors $p \in \mathcal{P}(G)$.

Proof. (i) Since the hyperplane $H(\alpha, t)$ contains the face G, the vector t lies in the space $X(G)$ that is orthogonal to affine space of the face G. The space $X(G)$ is generated by facet vectors $p \in \mathcal{P}(G)$. Moreover the vectors $p \in \mathcal{P}(G)$ generate a cone, where the vector t lies. Therefore the following representation $t = \sum_{p \in \mathcal{P}(G)} \mu_t(p)p$ holds, where $\mu_t(p) \geq 0$ for all $p \in \mathcal{P}(G)$. Any point $x \in G$ satisfies the equality $\langle t, x\rangle = \alpha(t)$ and $\langle p, x\rangle = \alpha(p)$ for all $p \in \mathcal{P}(G)$. Multiplying the last equalities for $p \in \mathcal{P}(G)$ by $\mu_t(p)$ and summing over all p, we obtain the wanted representation of $\alpha(t)$ through parameters $\alpha(p)$.

(ii) Since the number of vectors $p \in \mathcal{P}(G)$ is greater than dimension of the space $X(G)$ generated by $p \in \mathcal{P}(G)$, there is a linear dependence $\sum_{p \in \mathcal{P}(G)} \mu(p)p = 0$. As in the case (i), multiplying this dependence by $x \in G$, we obtain the wanted equality $\sum_{p \in \mathcal{P}(G)} \mu(p)\alpha(p) = 0$. □

If facets of a polytope are centrally symmetric, then they are organized in k-belts. Each k-belt of an n-polytope P is uniquely determined by a family of mutually parallel $(n-2)$ faces that are intersections of neighboring facets of this k-belt. The following assertion describes linear inequalities between $\alpha(p)$ related to 6-belts of a polytope.

Proposition 2. *Let p_1, p_2, p_3 be facet vectors of a 6-belt of a polytope P such that $p_3 = \mu_1 p_1 + \mu_2 p_2$, where $\mu_1, \mu_2 > 0$. Then this 6-belt determines the following inequalities between the three parameters $\alpha(p_i)$, $i = 1, 2, 3$,*

$$\mu_1 \alpha(p_1) + \mu_2 \alpha(p_2) \geq \alpha(p_3). \tag{3}$$

Proof. The facet vectors p_1, p_2, p_3 of a 6-belt of P lie in a 2-plane Π_2 that is orthogonal to mutually parallel $(n-2)$-faces of this belt. Therefore these three vectors are lnearly dependent. Let this dependence be $p_3 = \mu_1 p_1 + \mu_2 p_2$, where $\mu_1, \mu_2 > 0$. Any point $x \in H(\alpha, p_1) \cap H(\alpha, p_2)$ of the intersection of supporting hyperplanes of facets $F(p_1)$ and $F(p_2)$ is cut off from $P(\alpha)$ by the hyperplane $H(\alpha, p_3)$ that supports the facet $F(p_3)$. Hence, for this x, we have $\langle p_3, x\rangle \geq \alpha(p_3)$. Since $p_3 = \mu_1 p_1 + \mu_2 p_2$, this equality takes the form $\mu_1 \langle p_1, x\rangle + \mu_2 \langle p_2, x\rangle \geq \alpha(p_3)$. Obviously, the equalities $\langle p_1, x\rangle = \alpha(p_1)$ and $\langle p_2, x\rangle = \alpha(p_2)$ hold. These two equalities and the above inequality give the triangle inequality (3). □

Since the 3 facet vectors p_i for $i = 1, 2, 3$ are equivalent, each 6-belt gives 3 inequalities of type (3). If at least one of these inequalities, say (3), holds as equality, then the 6-belt is transformed into a 4-belt (p_1, p_2). In this case p_3 is not a facet vector, i.e. $p_3 \notin \mathcal{P}(P)$.

2. Parallelotopes

Proposition 1 gives nothing for a primitive parallelotope. If $P(\alpha)$ is a primitive or zonotopal parallelotope, then one can choose length of facet vectors p such that $\alpha(p) = \langle p, Dp\rangle$ for all $p \in \mathcal{P}$, where D is a positive definite matrix, and the type domain $\mathcal{D}(P)$ is determined by matrices D (see [1] and [6]).

If $P(\alpha)$ is a parallelotope, then each facet of it is centrally symmetric. Facets are special cases of *standard* faces of a parallelotope. Standard faces were defined by Dolbilin in [5]. Each standard face F is centrally symmetric. It is useful consider parallelotopes described by (1), where hyperplanes $H(\alpha, p)$ for all $p \in \mathcal{P}$ support standard faces of $P(\alpha)$. Vectors p of these hyperplanes may be find using item (i) of Proposition 1.

Let c_p be the center of a standard face $F(p)$ determined by a vector $p \in \mathcal{P}$. Then the parallelotope $P(\alpha)$ can be described by the inequalities (1) with $\alpha(p) = \langle p, c_p\rangle$.

Theorem 1. *A parallelotope $P(\alpha)$ is affinely equivalent to a Voronoi polytope if and only if there are lengths of facet vectors p such that $\alpha(p) = \langle p, Dp\rangle$ for some positive definite matrix D.*

Proof. Let $P(\alpha)$ be a parallelotope that is affinely equivalent to a Voronoi polytope. It is proved in [4] that then one can choose lengths of facet vectors p such that $c_p = Dp$, where D is a positive semi-definite matrix.

Conversely, if $\alpha(p) = \langle p, Dp\rangle$, then $c_p = Dp$. By [4], this means that $P(\alpha)$ is affinely equivalent to a Voronoi polytope. □

If $P = P(\alpha)$ is not primitive, then P has k-faces G such that $|\mathcal{P}(G)| > n - k$. In this case, the type domain $\mathcal{D}(P)$ is a face of the type domain of a primitive parallelotope.

Call a parallelotope P *rigid* if its type domain $\mathcal{D}(P)$ is one-dimensional. For a rigid parallelotope P, Proposition 1 allows to prove its rigidity, since a rigid parallelotope has sufficiently many k-faces G with $|\mathcal{P}(G)| > n - k$. In particular, one can show by this method that the Voronoi polytopes $P_V(D_4)$, $P_V(E_n)$, $P_V(E_n^*)$ for $n = 6, 7$, are rigid.

3. Zonotopes

In this section I give explicit expressions of $\alpha(p)$ for zonotopes and zonotopal parallelotopes.

Recall that a zonotope $Z(U)=\sum_{u\in U}b_uz(u)$ generated by a set of vectors $U\subset\mathbb{R}^n$ is the Minkowski sum of weighted segments

$$z(u)=\{x\in\mathbb{R}^n: x=\lambda u: -1\le\lambda\le 1\}.$$

Hence

$$Z(U)=\{x\in\mathbb{R}^n: x=\sum_{u\in U}\lambda_u u: \ -b_u\le\lambda_u\le b_u,\ u\in U\}. \tag{4}$$

Here $b_u\ge 0$ are non-negative weights for all $u\in U$. Below, $Z(U)$ denotes always the above sum. This is in a sense a "vertex description"of the zonotope $Z(U)$. Each vertex v of $Z(U)$ has a description $v=\sum_{u\in U}b_u(\pm u)$, where from two signs $\pm$ only one is taken. But we need a facet description of $Z(U)$.

Let $p\in\mathbb{R}^n$ be a vector. Define a subset $U_p\subseteq U$ as follows.

$$U_p=\{u\in U:\langle p,u\rangle=0\}.$$

The following Lemma 1 helps to find a facet description of $Z(U)$.

Lemma 1. *A shift of the zonotope $Z(U_p)$ is a face G of the zonotope $Z(U)$. The center of the face G is an end-point of the vector*

$$c_p=\sum_{u\in U-U_p}b_u\frac{\langle p,u\rangle}{|\langle p,u\rangle|}u. \tag{5}$$

The affine hyperplane

$$H_p=\{x\in\mathbb{R}^n:\langle p,x\rangle=\langle p,c_p\rangle\} \tag{6}$$

supports $Z(U)$ at the face $G=c_p+Z(U_p)\subset H_p$. In particular, if $U_p=U$, then $c_p=0$ and $G=Z(U)$, and if $U_p=\emptyset$, then G is a vertex that coincides with c_p.

Proof. Using (4), it is easy to see that c_p is a point of $Z(U)$. We show that the affine hyperplane H_p supports $Z(U)$, i.e. $\langle p,x\rangle\le\langle p,c_p\rangle$ for all $x\in Z(U)$. It is sufficient to verify these inequalities for vertices. We have $\langle p,v\rangle=\sum_{u\in U}b_u(\pm\langle p,u\rangle)$. Since $b_u\ge 0$ and $\langle p,u\rangle\in\{0,\pm1\}|\langle p,u\rangle|$ for all $u\in U$, the following inequality holds

$$\langle p,v\rangle\le\sum_{u\in U-U_p}b_u\frac{(\langle p,u\rangle)^2}{|\langle p,u\rangle|}=\langle p,c_p\rangle.$$

This inequality implies that the hyperplane H_p supports $Z(U)$, and $c_p+Z(U_p)=G$ is a face of $Z(U)$. □

Lemma 1 implies the following important

Theorem 2. *Let $U\subset\mathbb{R}^n$ be a set of vectors. Let $b_u\ge 0$ be non-negative weights for all $u\in U$. Let $\mathcal{P}$ be a set of vectors containing all facet vectors p of the zonotope $Z(U)=\sum_{u\in U}b_uz(u)$. Then the zonotope $Z(U)$ has the following description by linear inequalities*

$$Z(U)=\{x\in\mathbb{R}^n:\langle p,x\rangle\le\alpha_U(p)\ \textit{for all}\ p\in\mathcal{P}\}, \tag{7}$$

where

$$\alpha_U(p)=\sum_{u\in U-U_p} b_u\frac{(\langle p,u\rangle)^2}{|\langle p,u\rangle|}. \qquad (8)$$

Each inequality in (7) supports a face of $Z(U)$. □

Consider a zonotope $Z(U)$ generated by a unimodular set of vectors U. It is known (see, for example, [7], [3]) that a zonotope $Z(U)$ is a parallelotope if and only if vectors $u\in U$ are in proportion with vectors of a unimodular set of vectors. According to description of $Z(U)$ by (4), we can suppose that the set U is itself unimodular. Recall that a set is *unimodular* if each its vector has an integral representation in any its basic subset.

Let F be a facet of $Z(U)$ with its facet vector p which is, recall, orthogonal to F. The facet F is also a zonotope $Z(U_F)$, where $U_F=U_p$. The unimodular set U *represents* a *regular* matroid M_U (see any book on Matroid Theory, for example, [2]). The subset U_F represents a *copoint* of M_U. A definition of a copoint implies that the set of equalities $\langle p,u\rangle=0$ for all $u\in U_F$ determines uniquely up to a multiple a facet vector p. It is known, see, for example, [2], [3], that, for any copoint U_F of a regular matroid, lengths of its facet vectors p can be chosen such that they satisfy the following condition

$$\langle p,u\rangle\in\{0,\pm1\} \text{ for all } u\in U. \qquad (9)$$

It is important that fulfillment of this condition for all facet vectors is equivalent to unimodularity of the set U (see, for example, [2], [3]).

Let a vector p satisfies the condition (9). Then $|\langle p,u\rangle|\in\{0,1\}$, and, since $\langle p,u\rangle=0$ for $u\in U_p$, c_p can be written as

$$c_p=\sum_{u\in U} b_u(\langle p,u\rangle)u. \qquad (10)$$

Theorem 3. *Let $U\subset\mathbb{R}^n$ be a unimodular set of vectors. Let $b_u\ge 0$ be non-negative weights for all $u\in U$. Let $\mathcal{P}$ be a set of all facet vectors p of the zonotope $Z(U)=\sum_{u\in U} b_u z(u)$. Let facet vectors are chosen such that they satisfy conditions (9). Then the zonotope $Z(U)$ has the following description by linear inequalities*

$$Z(U)=\{x\in\mathbb{R}^n:\langle p,x\rangle\le f_U(p) \text{ for all } p\in\mathcal{P}\}, \qquad (11)$$

where

$$f_U(p)=\sum_{u\in U} b_u(\langle p,u\rangle)^2=\langle p,\sum_{u\in U} b_u(uu^T)p\rangle, \qquad (12)$$

is a positive semi-definite quadratic form on vectors $p\in\mathcal{P}$.

Proof. If p is a facet vector, then it satisfies conditions (9). Hence $|\langle p,u\rangle|=1$ for all $u\in U-U_p$ and the hyperplane (6) takes the form

$$H_p(f)=\{x\in\mathbb{R}^n:\langle p,x\rangle=f_U(p)\}. \qquad (13)$$

This implies the assertion of this theorem. □

By Theorem 1, Theorem 3 shows that the zonotopal parallelotope $Z(U)$ is affinely equivalent to a Voronoi polytope.

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Определение области типа параллелоэдра

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Ключевые слова: параллелоэдр, область типа, зонотоп

Любой выпуклый многогранник $P = P(\alpha)$ может быть описан системой линейных неравенств, определяемых векторами p и правыми частями $\alpha(p)$. Для фиксированного множества векторов p определяется область типа $\mathcal{D}(P_0)$ многогранника P_0, и в частности параллелоэдра P_0, как такое множество параметров $\alpha(p)$, что многогранники $P(\alpha)$ имеют тот же комбинаторный тип, что и P_0 для всех $\alpha \in \mathcal{D}(P_0)$. Во второй части статьи дается фасетное описание зонотопов и зонотопных параллелоэдров.

Статья публикуется в авторской редакции.

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