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THEORY OF COMPUTING

The Boltzmann distribution in the problem of rational choice by population of a patch under an imperfect information about its resources

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The problem of rational choice by the population of a patch containing energy (nutritive) resources is considered. This problem belongs to the theory of optimal foraging, which, in turn of, studies issues related to the behavior of the population when it leaves the patch or chooses the most suitable one. In order to define the optimal patch choice for population, a variational approach, based on the idea of the Boltzmann distribution is proposed. To construct the probability distribution the utility functions are used, that take into account factors that can influence the patch choice of a population: available information about the quality of patches, the energy utility of patches, the cost of moving to the patch, the cost of information about the quality of patches. The main goal of the paper is to investigate the influence of available information about the amount of resources, contained in patches, on a decision-making process generated by the foragers while a suitable patch choosing. The optimal rationality is determined in the cases taking into account the information cost, the average energy utility of all patches, the rationality depending on the patch. The conditions under which the population, with the lack of information, select the "poor" patch, in sense of its resources, are obtained. The latter provides a theoretical justification of experimental observations, according to which a population can choose a patch with worse quality. The obtained results have a general character and may be used not only in behavioral ecology but when constructing any decision making processes.

Keywords: Boltzmann distribution; rationality of choice; measure of awareness; information cost; utility function

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THEORY OF COMPUTING

Распределение Больцмана в проблеме рационального выбора популяцией участка при неполной информации о его ресурсах

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Рассматривается задача рационального выбора популяцией участка, содержащего энергетические (пищевые) ресурсы. Рассматриваемая задача относится к теории оптимального фуражирования, которая в свою очередь изучает вопросы, касающиеся поведения популяции, когда она покидает участок или выбирает наиболее подходящий. Для определения оптимального для популяции выбора участка предлагается вариационный подход, основанный на идее распределения Больцмана. Для построения распределения Больцмана вводятся функции полезности, которые учитывают факторы, способные повлиять на выбор популяции: имеющаяся информация о качестве участков, энергетическая полезность участков, затраты на перемещение к участку, стоимость информации о качестве участков. Основная цель статьи – исследовать влияние имеющейся информации о количестве ресурсов, содержащихся в участках, на процесс принятия решений, генерируемых популяцией при выборе подходящего участка. Оптимальная рациональность определяется с учетом стоимости информации, средней энергетической ценности всех участков, рациональности, зависящей от качества участка. Получены условия, при которых популяция при недостатке информации выбирает «бедный» участок в смысле энергетической ценности (ресурсов). Последнее дает теоретическое обоснование экспериментальным наблюдениям, согласно которым, популяция может выбрать участок худшего качества. Полученные результаты носят общий характер и могут быть использованы не только в поведенческой экологии, но и при построении любых процессов принятия решений.

Ключевые слова: распределение Больцмана; рациональность выбора; мера информированности; стоимость информации; функция полезности

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Introduction

The open problem of rationality and rational choice is one of the crucial challenges in artificial intelligence, reinforcement learning, computational neuroscience, behavioral ecology of animals, economics. It is important to understand the mechanisms of decision-making process under bounded resource, in particular, available information. R. Aumann, who received the nobel prize (2005) for investigations of conflict and cooperation, wrote [1] that the problem of rationality "…is perhaps the most challenging conceptual problem in the area today: to develop a meaningful formal definition of rationality in a situation in which calculation and analysis themselves are costly and/or limited". Moreover, the problem of bounded rationality becomes especially acute in connection with applications mentioned above (see, for example, [2], [3] and the references therein).

One of the applications of the rational choice theory is in the optimal foraging theory, which studies, in particular, the patch selection by population, that is most suitable for consumption of the resources contained in it. As it is generally accepted, the behavior of population aimed at maximizing the amount of consumed energy [4–8].

Within the framework of this theory, the concept of ideal free distribution (IFD) was proposed [9, 10]. According to the IFD, a population has perfect information about patches quality and is distributed between patches so as to maximize an energy consumption rate. But empirical observations show that the IFD model is not adequate to real patch selection processes. A population, under the lack of information about patch resources, may choose poor patches (see, for example, [11] and the references therein).

In order to overcome the disadvantages of the IFD concept, U. Dieckmann proposed an approach [11], based on the Boltzmann distribution and utility functions of patches. If p_i is the probability for a population to select the i-th patch, $i = 1, \ldots, n$, then, according to the Boltzmann distribution

$$p_i = \frac{e^{qU_i}}{\sum\limits_{j=1}^{n} e^{qU_j}}, \tag{1}$$

where U_i is the utility function corresponding to the i-th patch, q is non-negative constant, n is the number of patches.

It is worth to note that the Boltzmann distribution is one of the main notions of statistical physics [12]. Namely, consider a physical system which may be in one of the states $1, \ldots, n$ with energies $E_1, \ldots, E_n$, respectively. Then the probability p_i for the system to be in the i–th state is given by (1), where $U_i = E_i$, $q = -\dfrac{1}{kT}$, T is the temperature of the "large" external heat source, $k > 0$ is the constant.

The methods of statistical physics have become ubiquitous in mathematics and mathematical modeling. Such notions as the entropy, the Gibbs and the Boltzmann distributions, the thermodynamic potential, to name a few, play a significant role in ergodic theory [13], [14], machine learning (particularly, reinforcement learning) [15], [16], decision-making theory [17], information theory [18], etc.

This paper deals with a problem of decision making processes, on the basis of the Boltzmann distribution, in the foraging theory. As it is noted above, the foraging theory, originally, appeared as a branch of behavioral ecology. Its goal is to develop a theoretical base for description and explanation of animal behavior while food searching. Later on, the ideas of optimal foraging theory were used to formalize the processes of choice in different areas.For example, nowadays, we observe the appearance of such fields as information foraging [19] and robot foraging [20]. The latter permits us to use, in this paper, an abstract language without binding to the optimal foraging connected with the behavioral ecology. But, in order to make the presentation more pictorial, describing the decision-making process, along with the term "agent" we use the term "population".

In the paper, the authors continue their researches dealing with mathematical modeling in optimal foraging theory, namely, [21], [22], [23].

The main goal of the paper is to investigate the influence of available information about the amount of resources, contained in patches, on a decision-making process generated by the population while a suitable patch choosing. As one might expect, when decision makers have a limited amount of information, the rationality of their behavior is bounded [2, 17]. On the basis of the Boltzmann distribution, we investigate this problem in optimal foraging, namely, in the selection of suitable patch by population.

The paper is organized as follows. In the Introduction the main applications of the theory of optimal foraging and methods for studying its problems are presented. Section 1 describes the proposed variational principle for determination of the optimal choice. In Section 2 the influence of the imperfect information on the rationality of choice is analyzed. Section 3 deals with the case when the rationality q is not constant but depends on information. In the Conclusion the results of the study are summarized.

1. Optimal rationality

According to [11], the parameter q in (1) has the sense of "controlling the degree of optimality" in choosing a patch. In other words, q is the rationality of decision-making process, assuming $q \geqslant 0$. In what follows, we call the parameter q the rationality of patch selecting, or, more simply - the rationality. Such sense of q is justified by the following proposition.

Proposition 1. *Denote $U_{i_s} = \max_i \{U_i,\ i = 1, \ldots, n\}$, $s = 1, \ldots, k$. Then:*
$p_{i_s} \to \frac{1}{k}$ *as* $q \to \infty$;
$p_i \to 0$ *as* $q \to \infty$ *if* $i \in \{1, \ldots, n\} \setminus \{i_s,\ s = 1, \ldots, k\}$;
$p_i = \frac{1}{n}$ *if* $q = 0$.

Proof. We have the following presentation of p_i from (1)

$$p_i = \frac{1}{1 + \sum\limits_{j=1,\ j \neq i}^{n-1} e^{q(U_j - U_i)}}, \tag{2}$$

which implies the proof. □

Remark 1. The special case of Proposition 1, namely for $i = 1$, was formulated in [11].

The sense of Proposition 1 is as follows. The probability of choosing by an agent (population) the patch with the largest utility tends to its maximum value as q tends to infinity. If $k = 1$ (k is the number of maximum utilities, equal each other) then the maximum probability equals 1 and corresponds to the largest utility. Zero rationality, $q = 0$, means equiprobability of patch selection, or absolutely random choice. These facts confirm the sense of q as a measure of rationality.

Denote by V_i the amount of resources (energy) in the i-th patch, $V^* = \max\{V_j,\ j = 1, \ldots, n\}$. Obviously, the patch with the maximum utility U_{i_s} may not contain the maximum resource, i. e. $V_{i_s} \neq V^*$, that depends on the form of the utility function.

Let us consider V_i, $i = 1, \ldots, n$, as the values of a random variable V, and $p_i = P(V = V_i)$, determined by (1), is the probability for a population to choose the i−th patch. Denote by $E = E(V)$ and $D = D(V)$ the mathematical expectation and dispersion, respectively, of V. Let us consider p_i as the functions of $q : p_i = p_i(q)$, i. e. other parameters in p_i are supposed to be fixed. Then

$$E = E(V, q) = \sum_{i=1}^{n} V_i p_i(q) = \frac{1}{\sum\limits_{j=1}^{n} e^{qU_j}} \sum_{i=1}^{n} V_i e^{qU_i}.$$

In what follows, for convenience, $E(V, q)$ will be denoted as $E(q)$.

Remark 2. In what follows we write $E(\infty) = \lim_{q\to\infty} E(q)$. Moreover, if $E(q) < E(\infty)$ for all $q \geqslant 0$, we write that $\max E(q) = E(\infty)$.

It is easy to show that

$$\lim_{q\to\infty} E(q) = \frac{1}{k}\sum_{s=1}^{k} V_{i_s},$$

where i_s is determined in Proposition 1.

If $U_i = V_i$ and $V_i \neq V_j$, $i, j = 1, \ldots, n$, then $\lim_{q\to\infty} E(q) = V^*$. In such case

$$\max_q E(q) = V^*.$$

The latter argument and Proposition 1 motivate the following definition.

Definition 1. The rationality q^* is called optimal if

$$E(q^*) = \max_q E(q).$$

Definition 2. The choice under $q = q_1$ is more rational than that under $q = q_2$ if

$$E(q_1) > E(q_2).$$

Remark 3. Usually, $E(q)$ is considered as an average income when selecting corresponds to some distribution p_i, $i = 1, \ldots, n$. The latter justifies Definition 1 and provides one more confirmation of the interpretation of q as the value of patch selection rationality, given in [11].

Proposition 2.

$$\frac{dE(q)}{dq} = \sum_{i=1}^{n} V_i U_i p_i - \sum_{i=1}^{n} V_i p_i \cdot \sum_{i=1}^{n} U_i p_i, \tag{3}$$

where p_i are given by (1).

Proof. We have

$$\frac{dp_i(q)}{dq} = \frac{1}{\left(\sum\limits_{j=1}^{n} e^{qU_j}\right)^2} \cdot \left(U_i e^{qU_i} \cdot \sum_{j=1}^{n} e^{qU_j} - e^{qU_i} \cdot \sum_{j=1}^{n} U_j e^{qU_j}\right) = U_i p_i - p_i \cdot \sum_{i=1}^{n} U_j p_j,$$

hence

$$\frac{dE(q)}{dq} = \sum_{i=1}^{n} \frac{dp_i(q)}{dq} V_i = \sum_{i=1}^{n} V_i U_i p_i - \sum_{i=1}^{n} p_i V_i \cdot \sum_{i=1}^{n} p_i U_i.$$

□

Corollary 1. *Suppose $U_i = V_i$, $i = 1, \ldots, n$. Then*

$$\frac{dE(q)}{dq} = D(q),$$

where $D(q)$ is a dispersion of V.

Hence, if $U_i = V_i$, $i = 1, \ldots, n$, then $E(q)$ is monotonically increasing with respect to q, and $\max_q E(q) = \lim_{q\to\infty} E(q) = E(\infty) = V^*$. But if there exists k such that $U_k \neq V_k$ then the monotone increasing of $E(q)$ cannot be guaranteed, and $q = \infty$ may not be the optimal rationality.

2. Bounded rationality

Let us consider the influence of imperfect information or the lack of it on the rationality of choice. Such case may be classified as the choice under bounded rationality. In [11] the following utility function U_i, characterizing the i-th patch from the point of view of population, was proposed

$$U_i = I_i V_i + (1 - I_i)\overline{V} - T(d_i), \tag{4}$$

where I_i is the population measure of the patch i awareness, $I_i \in [0, 1]$, V_i is the amount of food (energy) resource containing in the i-th patch, $\overline{V} = \gamma_1 V_1 + \ldots + \gamma_n V_n$ is the average utility of the patches for a population, $\gamma_1 + \ldots + \gamma_n = 1$, $\gamma_i \geqslant 0$, $T(d_i)$ is the cost of moving to the i-th as a function of the distance d_i to it.

As it was noted in Introduction, we consider the general problem of rational choice. Therefore, we do not use the distance parameter d_i. The aim of this paper is to give an approach to modeling the optimal rational choice. Thus we consider the following form of U_i

$$U_i = I_i V_i + (1 - I_i)\overline{V} - C_i(q), \tag{5}$$

where $C_i(q)$ is the cost of information about the patch i. It is worth to note the dependence of the cost on rational parameter q, that models the influence of rational behavior on the cost. Detailed description of C_i will be given below. In what follows, we, on the basis of (5), find the optimal rationality q^* in the sense of proposed approach $q^* = \arg\max_q E(q)$ and investigate the influence of the imperfect information on rationality of choice.

2.1. Free information

Let us consider the following utility functions:

$$U_i = I_i V_i,\ I_i \in [0; 1].$$

Such form of U_i underlines the fact that the utility of the $i-$th patch depends on available information about its resources. The latter implies the possibility of the situation when $V_i > V_j$ but $U_i < U_j$. The following proposition refines this reasoning for two patches.

Proposition 3. *Assume $n = 2$, $U_i = I_i V_i$, $i = 1, 2$, $V_2 > V_1$.*
If $I_2 V_2 > I_1 V_1$ then $\max_q E(q) = E(\infty) = V_2$.
If $I_2 V_2 < I_1 V_1$ then $\max_q E(q) = E(0) = \dfrac{V_1 + V_2}{2}$.
If $I_2 V_2 = I_1 V_1$ then $E(q) = E(0)$ for any $q > 0$.

The proof directly follows from (2) and (3). The Proposition 3 shows that the optimal rationality depends on the available information. The lack of information implies that the average income is not the largest: $\dfrac{V_1 + V_2}{2} < V_2$ in the second case.

Remark 4. It is worth to note that in the second case the choice of any patch is equally probable: $p_1 = p_2 = \dfrac{1}{2}$. This result gives an explanation of the fact that in practice a population makes a choice which is not agreeable with IFD conception. Really, the IFD asserts that the choice of population is optimal if $q = \infty$ which implies that $E(\infty) = V_1 < \dfrac{V_1 + V_2}{2}$. But our variational approach gives the value $E(0) > E(\infty)$ for the absolutely random choice confirmed by practice [11].

Remark 5. The result, analogous to Proposition 3, for $n > 2$, may be easily obtained though without such clarity as for $n = 2$.

2.2. Average utility

Let us consider the utility function taking into account the average utility of the patches $\overline{V} = \gamma_1 V_1 + \ldots + \gamma_n V_n$ about patches

$$U_i = I_i V_i + (1 - I_i)(\gamma_1 V_1 + \ldots + \gamma_n V_n), \tag{6}$$

where $\gamma_1 + \ldots + \gamma_n = 1$, $\gamma_1 \geqslant 0$. It follows from (6) that if $I_i = 1$, i. e. an agent (population) has the perfect (full) information about the resources V_i of i-th patch, then the average utility is being removed because of it uselessness. And vice-versa, if $I_i = 0$, i. e. an agent has no information about V_i, utility U_i is being replaced by average utility of patches.

Proposition 4. *Assume* $n = 2$, $V_2 > V_1$, U_i, $i = 1, 2$, *is determined by* (6). *Then*

$$\frac{dE(q)}{dq} > 0.$$

Proof. Consider $U_2 - U_1 = (I_2 V_2 + (1 - I_2)\overline{V}) - (I_1 V_1 (1 - I_1)\overline{V}) = (V_2 - \overline{V})I_2 + (\overline{V} - V_1)I_1 > 0$. Then, taking into account that $V_2 > V_1$ and (2), we finish the proof. □

Corollary 2. *Under the assumptions of Proposition 4,* $\max_q E(q) = E(q^*) = E(\infty) = V_2$.

Remark 6. It is interesting to compare the conclusions of Propositions 3 and 4. The addition of average income, $\overline{V}$, (Proposition 4), strengthens the population in its solution to choice a "rich" patch ($V_2 > V_1$): it is chosen always. If $U_i = I_1 V_i$ then the rich patch is being chosen only under the condition $I_2 V_2 > I_1 V_1$ (Proposition 3).

2.3. Information cost

Assume that the utility function of the i-th patch, without information cost, is $U_i = V_i I_i$, where $I_i \in [0, 1]$ is an agent measure of the awareness of the patch i. Suppose an agent has to pay some cost for information about the amount of resources V_i in patches. Denote by $\beta_i \in [0, V_i]$ the price of the unit of information I_i about V_i. Hence, $\beta_i I_i$ is the cost of information I_i.

Let us assume that an agent's rational behavior diminishes the information cost to the value $\beta_i I_i f_i(q)$, where $f_i(q) \in [0, 1]$ for $q \geqslant 0$.

Therefore, the utility function of the i-th patch has the form

$$U_i = I_i V_i - \beta_i I_i f_i(q).$$

Then we can easily obtain that

$$\frac{dE(q)}{dq} = \sum_{i=1}^{n} V_i p_i (U_i + qU_i') - \sum_{i=1}^{n} V_i p_i \cdot \sum_{i=1}^{n} p_i (U_i + qU_i'),$$

where $U_i' = \dfrac{dU_i}{dq} = -\beta_i I_i f_i'(q)$. To make an analysis of $E(q)$ more pictorial, let us introduce the following assumptions on $f_i(q)$ and consider the case $n = 2$. Assume that

$f_1 = f_2 = f$, which means the equality of agent's possibilities to diminish information cost for all patches;

$f(0) = 1$, which means the absence of influence of an "irrational" ($q = 0$) agent on the information cost;

$f(q) \in [0, 1]$ for $q \geqslant 0$.

It is easy to show that

$$E(q) = \frac{V_1 - V_2}{1 + e^{q(U_2 - U_1)}} + V_2, \tag{7}$$

and, hence

$$\frac{dE(q)}{dq} = \frac{(V_2 - V_1)e^{q(U_2-U_1)}}{(1 + e^{q(U_2-U_1)})^2} \cdot (U_2 - U_1 - q(U_2' - U_1')).$$

Denote

$$A = I_2V_2 - I_1V_1, \quad B = I_2\beta_2 - I_1\beta_1.$$

Then, $U_2 - U_1 - q(U_2' - U_1') = A - Bf(q) + qBf'(q) = B\left(\frac{A}{B} - (f(q) - qf'(q))\right)$. Denote $g(q) = f(q) - qf'(q)$. Then

$$\frac{dE(q)}{dq} = \frac{(V_2 - V_1)e^{q(U_2-U_1)}}{(1 + e^{q(U_2-U_1)})^2} \cdot B\left(\frac{A}{B} - g(q)\right). \tag{8}$$

Hence, we can formulate the following obvious proposition.

Proposition 5. *Assume $V_2 > V_1$, $f \in C^1[0, \infty)$. Then*

$\max_q E(q) = E(q^*)$ if $\frac{A}{B} > 1$ and there exists the unique $q^* > 0$ such that $g(q^*) = \frac{A}{B}$, $g(q) < \frac{A}{B}$ for $q \in (q^*, \infty)$;

$\max_q E(q) = E(\infty)$ if $\frac{A}{B} - g(q) > 0$ for $q \geqslant 0$;

$\max_q E(q) = E(0) = \frac{V_1 + V_2}{2}$ if $\frac{A}{B} - g(q) < 0$ for $q \geqslant 0$.

The proof directly follows from (8). Proposition 4 shows the influence of $f(q)$, which may be considered as the rational cost behavior, on the optimal rationality.

Example 1. $f_i(q) = \frac{1}{1 + \alpha_i q}$, $\alpha_i \geqslant 0$, where α_i is the measure of influence of q on the cost of information I_i.

We have

$$\frac{dE(q)}{dq} = \frac{(V_2 - V_1)e^{q(U_2-U_1)}}{\left(1 + e^{q(U_2-U_1)}\right)^2}\left(A - \left(\frac{\beta_2 I_2}{1 + \alpha_2 q} - \frac{\beta_1 I_1}{1 + \alpha_1 q}\right) - q\left(-\frac{\alpha_2\beta_2 I_2}{(1 + \alpha_2 q)^2} + \frac{\alpha_1\beta_1 I_1}{(1 + \alpha_1 q)^2}\right)\right).$$

Assume, for simplicity of transformations, that $\alpha_1 = \alpha_2 = 1$. Denote $x = \frac{1}{1+q}$. From the equality above, it is easy to obtain

$$\frac{dE(q)}{dq} = \frac{(V_2 - V_1)e^{q(U_2-U_1)}}{\left(1 + e^{q(U_2-U_1)}\right)^2}(A - B \cdot x^2). \tag{9}$$

Thus, we can obtain the following result, a special case of Proposition 4.

Proposition 6. *Assume $V_2 > V_1$. Then*

1. *$\frac{dE(q)}{dq} = 0$ for $q = q^*$ if and only if $\frac{B}{A} > 1$, and with this $q^* = \sqrt{\frac{\beta_2 I_2 - \beta_1 I_1}{I_2V_2 - I_1V_1}} - 1$.*
2. *If $A > 0,\ B > 0,\ A < B$, then $E(q^*) = \min_q E(q)$, $\lim\limits_{q\to\infty} E(q) = V_2$.*
3. *If $A < 0,\ B < 0,\ B < A$, then $E(q^*) = \max_q E(q)$, $\lim\limits_{q\to\infty} E(q) = V_1$.*

The proof easily follows from (9), (7) and $U_2 - U_1 = A - \dfrac{B}{1+q}$.

Figures 1 and 2 illustrate cases 2 and 3 of the Proposition 6 respectively. Figures 3 and 4 illustrate behaviour of function $h(q) = A - B\dfrac{1}{(1+q)^2}$ depending on the parameters V_i, I_i, β_i, $i = 1, 2$.

At the same time, figures 1 and 3, as well as 2 and 4, reflect the influence of the function $h(q)$ on the sign of the derivative (9) and, accordingly, it's influence on the optimal value of the mathematical expectation (7).

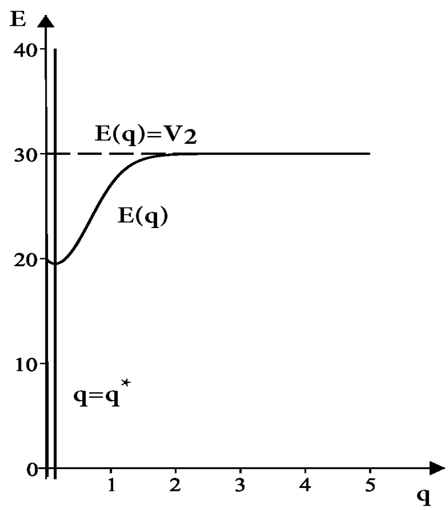

Fig. 1. Expected value $E(q)$ with $V_1 = 10$, $V_2 = 30$, $I_1 = 0,4$, $I_2 = 0,7$, $\beta_1 = 25$, $\beta_1 = 5$.

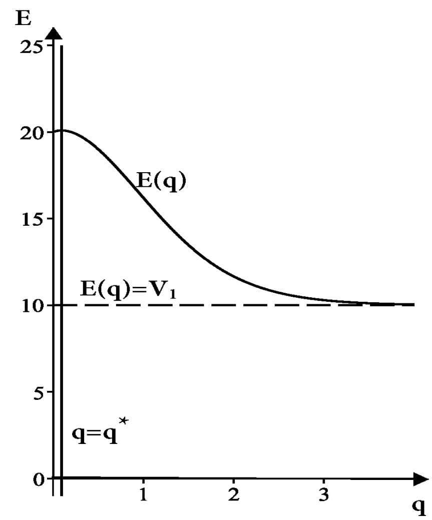

Fig. 2. Expected value $E(q)$ with $V_1 = 10$, $V_2 = 30$, $I_1 = 0,2$, $I_2 = 0,8$, $\beta_1 = 20$, $\beta_1 = 8$.

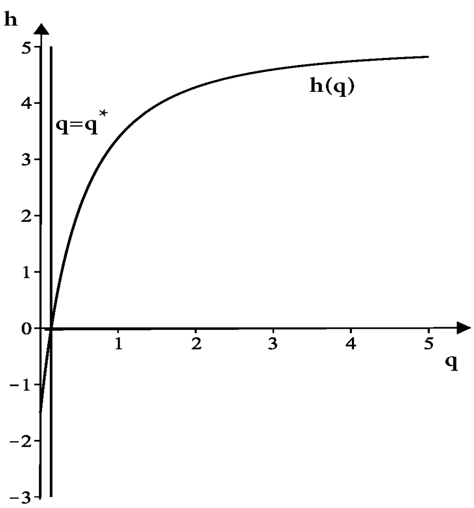

Fig. 3. $h(q)$, $V_1 = 10$, $V_2 = 30$, $I_1 = 0,4$, $I_2 = 0,7$, $\beta_1 = 25$, $\beta_1 = 5$.

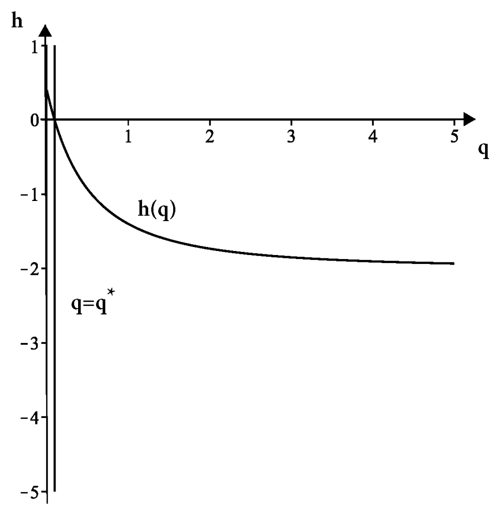

Fig. 4. $h(q)$, $V_1 = 10$, $V_2 = 30$, $I_1 = 0,2$, $I_2 = 0,8$, $\beta_1 = 20$, $\beta_1 = 8$.

3. Variable rationality

1. It is naturally to suppose that rationality of selection is the function of information, available to population, about patches. Hence, in what follows, we set $q = q(I_1, \ldots, I_n)$.

In order not to complicate transformations below, let us consider the case of two patches, $q = q(I_1, I_2)$ and $U_i = I_i V_i$, $i = 1, 2$.

We have

$$p_i = \frac{e^{q(I_1,I_2)U_i}}{\sum\limits_{j=1}^{2} e^{q(I_1,I_2)U_j}},$$

and $E = E(I_1, I_2)$.

Suppose, q is a differentiable function. Denote $g = g(I_1, I_2) = q(I_2V_2 - I_1V_1)$, $q_i = \frac{\partial q}{\partial I_i}$, $g_i = \frac{\partial g}{\partial I_i}$, $i = 1, 2$. Then

$$\frac{\partial E}{\partial I_i} = (V_1 - V_2)\frac{e^g g_i}{(1 + e^g)^2},$$

where

$$g_1 = q_1(I_2V_2 - I_1V_1) - qV_1, \quad g_2 = q_2(I_2V_2 - I_1V_1) + qV_2.$$

If $(I_1,\ I_2)$ is the stationary point of $E(I_1, I_2)$ then

$$\begin{cases} g_1(I_1, I_2) = q_1(I_2V_2 - I_1V_1) - qV_1 & = & 0, \\ g_2(I_1, I_2) = q_2(I_2V_2 - I_1V_1) + qV_2 & = & 0. \end{cases}$$

The necessary condition for solvability of this system with respect to I_1, I_2 is the following equality

$$V_1\frac{\partial q}{\partial I_2} + V_2\frac{\partial q}{\partial I_1} = 0.$$

Therefore, if $\frac{\partial q}{\partial I_2} \cdot \frac{\partial q}{\partial I_1} > 0$ then $E(I_1, I_2)$ has no points of extremum. It is impossible to obtain interesting results about optimality of E without specification of $q = q(I_1, I_2)$.

2. Now, consider the following case

$$p_i = \frac{e^{q_iU_i}}{\sum\limits_{j=1}^{n} e^{q_jU_j}},$$

i. e. the rationality of choice depends on the patch. For example, an agent uses different methods of acquisition of information about patch resources. To illustrate the influence of the lack of information on the rationality of choice, let us consider a special case: $n = 2$, $q_1 = q(I)$, $q_2 = q(\alpha I)$, $I \in [0, 1]$, $\alpha \in [0, 1)$, $U_1 = I_1V_1 = IV_1$, $U_2 = I_1V_2 = \alpha IV_2$. Hence

$$p_1 = \frac{e^{q(I)IV_1}}{e^{q(I)IV_1} + e^{q(\alpha I)\alpha IV_2}}, \quad p_2 = \frac{e^{q(\alpha I)IV_1}}{e^{q(I)IV_1} + e^{q(\alpha I)\alpha IV_2}}.$$

We consider E as the function of $I : E = E(I)$. Let us assume that $q(0) = 0$. Really, the rationality of choice, obviously, equals zero if an agent has no information about a patch. Moreover, if $q = 0$ then it is equally probable to choose any patch, as it was noted above.

From (7)

$$E(I) = (V_1 - V_2) \cdot \frac{1}{1 + e^{g(I)}} + V_2,$$

where $g(I) = q(\alpha I)\alpha IV_2 - q(I)IV_1$. Then

$$E'(I) = (V_2 - V_1) \cdot \frac{e^{g(I)}}{(1 + e^{g(I)})^2} \cdot g'(I),$$

where $g'(I) = \frac{dq(u)}{du}|_{u=\alpha I}\alpha^2V_2I + q(\alpha I)\alpha V_2 - (q'(I)V_1I + q(I)V_1)$. We have an obvious proposition.

Proposition 7. *Assume* $V_2 > V_1$.
If $g'(I) > 0$, *for any* $I \in (0; 1)$, *then* $\max_I E(I) = E(1)$.
If $g'(I) < 0$, *for any* $I \in (0; 1)$, *then* $\max_I E(I) = E(0)$.

Let us discuss this result. We see that the maximum of available information, $I = 1$, does not guarantee the maximum value of $E(I)$. In particular, an additional, sufficient, condition is required, namely $g'(I) > 0$. The second case, $g'(I) < 0$, provides the sufficient condition of absolutely irrational behavior. An agent is unable to dispose of available information.

Conclusion

In this paper the problem of rational for population patch choice was considered. The approach based on the idea of the Boltzmann distribution was proposed to define the optimal patch choice. Utility functions were used to construct the Boltzmann distribution. The methods for analysis of the rationality of the patch choice were proposed: analysis of the mathematical expectation as a function depending on the parameter q (Optimal rationality, Bounded rationality) and as a function depending on the measure of the patch awareness I (Variable rationality). The influence of available information about the amount of resources contained in the patch on the decision-making process of patch choice was investigated.

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