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Method of the Logistic Function for Finding Analytical Solutions of Nonlinear Differential Equations

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The method of the logistic function is presented for finding exact solutions of nonlinear differential equations. The application of the method is illustrated by using the nonlinear ordinary differential equation of the fourth order. Analytical solutions obtained by this method are presented. These solutions are expressed via exponential functions.

Introduction

Nonlinear differential equations and their solutions play an important role at description of physical and other processes and as a result we can observe a large number of publications in this area.

There are a lot of methods for finding exact solutions of nonlinear equations. For example, the inverse scattering transform [1], the dressing method [2, 3], the Hirota method [4], the group methods [5] and some others demonstrate a lot of advantages in the case of exactly solvable nonlinear differential equations.

However, most of these methods do not give any new results in application to nonlinear nonintegrable equations. In this case researchers often use ansätze methods.

The list of these approaches is extensive. Let us mention: the singular manifold method [6], the trial function method [7], the tanh-expansion method [8–10], the simplest equation method [11–15], the G'/G - expansion method [16–18] and the F-expansion method [19–21].

Modern computer algebra systems like Mathematica and Maple play the main role in the application of these methods. Using these powerful programs a researcher can make cumbersome analytical calculations in a short period of time.

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It is well known that any expression containing exponents, trigonometric or hyperbolic functions can be rewritten in various forms. In the case of large expressions the equivalence of these forms is not obvious. Using the computer algebra programs many researchers find exact solutions of nonlinear differential equations but do not analyze the obtained results. They often suppose that a new ansätze can give new solutions, but they are often wrong. A lot of such examples and the list of common errors is given in the work [22].

In this paper we demonstrate the most simple method for finding solitary wave solutions of nonlinear differential equations with the application of the logistic function. The logistic function (the sigmoid function) is determined by the following formula [24–27]

$$Q(z) = \frac{1}{1+e^{-z}}, \tag{1}$$

where z is independent variable on the complex plane. We see that the logistic function has the pole of the first order on complex plane. This function can be used for finding exact solutions of nonlinear differential equations [27,28]. Other variants of this approach without the logistic function were used in some papers before (see, for a example [6,7,11,12,29]).

One can see that the logistic function is the solution of the first order differential equation called the Riccati equation [27,28]

$$Q_z - Q + Q^2 = 0. \tag{2}$$

The logistic function (1) can be presented taking the hyperbolic tangent into account because of the following formula

$$\frac{1}{1+e^{-z}} = \frac{1}{2}\tanh\left(\frac{z}{2}\right) + \frac{1}{2}. \tag{3}$$

However the logistic function is more convenient for finding exact solutions as it has been illustrated in recent papers [27,28].

Let us show that the general solution of the Riccati equation can be expressed via the logistic function. The Riccati equation takes the form

$$y_z = a\,y^2 + b\,y + c, \tag{4}$$

where a, b and c are arbitrary constants.

It is easy to obtain that the general solution of equation (4) can be written by means of formula

$$y = B - \frac{2\,B+b}{a}\,Q(z) \qquad z = \frac{z' - z_0}{2\,B+b}, \tag{5}$$

where z_0 is an arbitrary constant, B is defined via constants a, b and c from the algebraic equation

$$a\,B^2 + b\,B + c = 0. \tag{6}$$

So, the logistic function is the solution of the Riccati equation to within transformations (5).

The aim of this paper is to to use the logistic function to look for the analytical solutions of the generalized nonlinear wave equation in the form:

$$u_t + \alpha\, u^n\, u_x - \delta\,(u^m\, u_x)_x + u_{xx} + \sigma\, u_{xxx} + u_{xxxx} = 0, \tag{7}$$

where α, δ and σ are parameters of the equation, m and n are integer. This equation is applied for description of many nonlinear waves.

In the case of $n = 1$ and $\delta = 0$ equation (7) takes the form

$$u_t + \alpha\, u\, u_x + u_{xx} + \sigma\, u_{xxx} + u_{xxxx} = 0. \tag{8}$$

Nonlinear evolution equation (8) has been studied by many authors from various points of view. This equation has drawn much attention not only because it is interesting as a simple one-dimensional nonlinear evolution equation including effects of instability and dissipation but it also is important for description of engineering and scientific problems. Equation (8) was used in work [30] for explanation of the origin of persistent wave propagation through medium of reaction-diffusion type. In paper [31] equation (8) was derived for description of the nonlinear evolution of the disturbed flame front. We can encounter the application of equation (8) for studying of motion of a viscous incompressible fluid flowing down an inclined plane [32–34]. Mathematical modeling of dissipative waves in plasma physics by means of equation (8) was presented in [35]. Elementary particles as the solutions of the Kuramoto—Sivashinsky equation were studied in [36]. Equation (8) also can be used for description of long nonlinear waves in viscoelastic tube [37].

The exact solutions of the Kuramoto—Sivashinsky equation are well known. The solutions of Eq. (8) at $\sigma = 0$ were first found by Kuramoto [30]. Later Eq. (8) and its generalizations were considered many times. For example, the exact solutions of these equations were obtained and re-discovered in works [6, 7, 38–46].

In contrast to the Kuramato-Sivashinsky equation (8), equation (7) was not studied at various values m and n. Using the traveling wave solution

$$u(x,t) = w(z), \qquad z = k\,x + \omega\,t, \tag{9}$$

equation (7) can be written as the following:

$$k^4\, w_{zzz} + \sigma\, k^3\, w_{zz} + k^2\, w_z - \delta\, k^2\, w^m\, w_z + \frac{\alpha\, k}{n+1}\, w^{n+1} + \omega\, w + C_1 = 0. \tag{10}$$

To study equation (10) we apply the Painlevé test using three steps of the Kovalevskaya method.

At the first step we determine the first member of the expansion of the general solution in the Laurent series. This is accomplished by the means of the substitution

$$w = \frac{a_0}{z^p} \tag{11}$$

into the leading members of equation (10).

The Painlevé analysis of the equation can be continued further if the value of the power p will be integer.

At the second step we determine the Fuchs indices of the expansion for solution in the Laurent series. For this purpose we use the formula

$$w = \frac{a_0}{z^p} + b\, z^{j-p}. \tag{12}$$

Here b is the coefficient of the expansion for the solution in the Laurent series which cannot be found. The Fuchs indices can be found by means of a substitution of expression (12) into the equation with the leading members again and equating the expression with the first power β to zero.

At the third step we substitute the expansion of the general solution in the Laurent series with undermined coefficients in the form

$$w(z) = \frac{a_0}{z^p} + \frac{a_1}{z^{p-1}} + \frac{a_2}{z^{p-2}} + \dots \tag{13}$$

into equation (10). At this step we check the existence of the arbitrary constants in the Laurent series of the general solution of equation studied. In the case when there are three arbitrary constants in the Laurent series we have the necessary condition for integrable nonlinear differential equation. As this takes place we can obtain three arbitrary constants in the expansion: two arbitrary coefficients in (13) and constant z_0 which can be added to the variable z.

Considering results of the expansion for solution in the Laurent series we can draw conclusions regarding exact solutions.

The aim of this work is to analyze equation (10) and to find its exact solutions using the logistic function.

1. Painlevé analysis and exact solution of equation at $m = 1$ and $n = 1$

Let us consider equation (10) at $m = 1$ and $n = 1$. This equation was derived for description of survace waves in a convecting fluids [47–49]. Using the traveling wave solution we can write equation (10) in the form:

$$k^4\, w_{zzz} + \sigma\, k^3\, w_{zz} + k^2\, w_z - \delta\, k^2\, w\, w_z + \frac{\alpha\, k}{2}\, w^2 + \omega\, w + C_1 = 0. \tag{14}$$

Without loss of the generality let us take the value of parameter $\delta = 6$ in equation (14).

Substituting w from (11) into the equation with the leading members

$$k^2\, w_{zzz} - 6\, w\, w_z = 0 \tag{15}$$

we find the values a_0 and p: $(a_0,\, p)$=$(2\, k^2,\, 2)$. The general solution of equation (14) has the pole of the second order.

Substituting

$$w = \frac{2\, k^2}{z^2} + b\, z^{j-2} \tag{16}$$

into equation (15) we obtain the Fuchs indices in the form

$$j_1 = -1, \qquad j_2 = 4, \qquad j_3 = 6. \tag{17}$$

The index $j_1 = -1$ corresponds the arbitrary constant z_0, but for the further consideration we have to substitute the expansion of the solution in the Laurent series in the form

$$w = \frac{2\,k^2}{z^2} + \frac{a_1}{z} + a_2 + a_3\,z + a_4\,z^2 + a_5\,z^3 + a_6\,z^4 + \ldots \tag{18}$$

into equation (14). Equating expressions at different powers of z to zero, we have the following system of equations:

$$a_1 = -\frac{k}{15}\,(\alpha + 6\,\sigma)\,, \tag{19}$$

$$a_2 = \frac{23\,\alpha\,\sigma}{900} - \frac{\sigma^2}{150} + \frac{\alpha^2}{225} + \frac{1}{6}, \tag{20}$$

$$a_3 = \frac{\omega}{6\,k^2} - \frac{16\,\alpha^2\sigma}{3375\,k} - \frac{\alpha\,\sigma^2}{1500\,k} - \frac{7\,\alpha^3}{9000\,k} - \frac{\alpha}{36\,k} - \frac{\sigma^3}{750\,k}, \tag{21}$$

$$k\,(\alpha + \sigma)\left(\alpha^3 k + 6\,\alpha^2 k\sigma + 36\,\alpha\,k + 216\,\omega\right) = 0, \tag{22}$$

$$\begin{gathered}
a_5 = -\frac{\alpha\,\sigma^2}{2700\,k^3} + \frac{53\,\alpha^2\sigma}{16200\,k^3} + \frac{\alpha^3}{1800\,k^3} + \frac{C_1}{6\,k^4} + \frac{11\,\sigma\,a_4}{15\,k} + \\
+ \frac{2\,\alpha\,a_4}{5\,k} + \frac{\omega}{36\,k^4} + \frac{\alpha}{432\,k^3} + \frac{11\,\alpha\,\sigma^4}{90000\,k^3} + \frac{53\,\omega\,\alpha\,\sigma}{2700\,k^4} + \\
+ \frac{4781\,\alpha^3\sigma^2}{9720000\,k^3} + \frac{167\,\alpha^5}{12150000\,k^3} - \frac{\sigma^5}{112500\,k^3} - \frac{\omega\,\sigma^2}{450\,k^4} + \\
+ \frac{\omega\,\alpha^2}{300\,k^4} + \frac{\alpha^2\sigma^3}{45000\,k^3} + \frac{799\,\alpha^4\sigma}{4860000\,k^3},
\end{gathered} \tag{23}$$

$$\begin{gathered}
\frac{763\,\alpha^6}{24300000} + \frac{1957}{27000}\,\frac{\omega\,\alpha^2\sigma}{k} + \frac{17}{3}\,k^2 a_4\,\alpha\,\sigma + \frac{461}{2250}\,\frac{\alpha\,\omega\,\sigma^2}{k} - \frac{7}{250}\,\frac{\omega\,\sigma^3}{k} + \\
+ \frac{7\,\alpha\,C_1}{15\,k} + \frac{169}{162000}\,\alpha^4 + 8\,k^2 a_4\,\sigma^2 + \frac{4241}{8100000}\,\alpha^5\sigma + \frac{97}{75000}\,\alpha\,\sigma^5 - \\
- \frac{\omega\,\alpha}{30\,k} + \frac{169}{27000}\,\frac{\omega\,\alpha^3}{k} + \frac{8\,\alpha^2\sigma^4}{16875} + \frac{14059}{4860000}\,\alpha^4\sigma^2 + k^2 a_4\,\alpha^2 + \\
+ \frac{3\,\sigma\,\omega}{10\,k} + \frac{9\,\sigma\,C_1}{5\,k} + \frac{461}{13500}\,\alpha^2\sigma^2 - \frac{7}{1500}\,\alpha\,\sigma^3 - \frac{\omega^2}{3\,k^2} + \frac{\alpha\,\sigma}{40} - \\
- \frac{\alpha^2}{360} - \frac{\sigma^6}{9375} + \frac{1957}{162000}\,\alpha^3\sigma + \frac{8639}{1620000}\,\alpha^3\sigma^3 = 0.
\end{gathered} \tag{24}$$

Considering system of equations (19) — (24) one can see that equation (14) does not pass the Painlevé test because (22) and (24) are not identically equal to zero. One can note that the coefficient a_4 can be an arbitrary constant. In this case we can obtain the exact solution with two arbitrary constants. In another case we can find the exact solution which is expressed via the equation of the first order and has only one arbitrary constant. In this case we can find the dependence ω on parameters of equation (14) in the form

$$\omega = -\frac{\alpha^3 k}{216} - \frac{\alpha^2 k\sigma}{36} - \frac{\alpha\, k}{6}\,. \tag{25}$$

Substituting (25) into (24) we obtain an equation with the coefficient a_4 and the parameters of equation (14). One can find from this equation the coefficient a_4 or parameter C_1 or other parameters. So, there is the expansion of solution (14) in the Laurent series but with only one or two arbitrary constants.

We use the method of the logistic function [23] for finding the exact solution of equation (14). For this purpose we look for the solution of equation (14) in the form

$$w = A_0 + A_1\, Q(z) + A_2\, Q(z)^2, \tag{26}$$

where A_0, A_1 and A_2 are coefficients which are found at comparison with the Laurent series of the solution for equation (14), $Q(z)$ is the function in the form

$$Q(z) = \frac{1}{1+e^{-z+z_0}}, \tag{27}$$

z_0 is an arbitrary constant.

The solutions $w(z)$ of equation (14) can be written in the form

$$w_1(z) = \frac{1}{6} + \frac{13\,\alpha\,\sigma}{900} - \frac{\sigma^2}{25} + \frac{19\,\alpha^2}{5400} + \left(\frac{\alpha^2}{450} + \frac{2\,\alpha\,\sigma}{75} + \frac{2\,\sigma^2}{25}\right) Q(z)^2, \tag{28}$$

and

$$\begin{gathered} w_2(z) = \frac{1}{6} + \frac{37\,\alpha\,\sigma}{900} + \frac{\sigma^2}{25} + \frac{31\,\alpha^2}{5400} - \left(\frac{\alpha^2}{225} + \frac{4\,\alpha\,\sigma}{75} + \frac{4\,\sigma^2}{25}\right) Q(z) + \\ + \left(\frac{\alpha^2}{450} + \frac{2\,\alpha\,\sigma}{75} + \frac{2\,\sigma^2}{25}\right) Q(z)^2, \end{gathered} \tag{29}$$

where z can be written as

$$z_{1,2} = \mp\frac{(\alpha+6\sigma)}{30}\,x \pm \frac{\alpha}{6480}\,(\alpha+6\,\sigma)\left(6\,\alpha\,\sigma + 36 + \alpha^2\right)\,t. \tag{30}$$

Solutions $w_1(z)$ and $w_2(z)$ satisfy equation (14) at additional condition

$$C_1 = -\frac{\alpha\,(\alpha+6\,\sigma)}{1749600000}\left(19\,\alpha^2 + 900 + 78\,\alpha\,\sigma - 216\,\sigma^2\right)\left(31\,\alpha^2 + 900 + 222\,\alpha\,\sigma + 216\,\sigma^2\right). \tag{31}$$

There also is a solution expressed via the Weierstrass function. This solution of equation (14) can be written as

$$w(z) = \frac{1}{6} + 2\,k^2\,\wp(z,\,g_2,\,g_3), \tag{32}$$

where

$$z = k\,x - \frac{\alpha\,k}{6}\,t, \qquad \sigma = -\frac{\alpha}{6}, \qquad g_2 = \frac{1}{12\,k^4} - \frac{6\,C_1}{\alpha\,k^5}. \tag{33}$$

So, we have found the solution of equation (14) with two arbitrary constants k and g_3 as we predicted before. This solution is realized at condition on parameters α and σ. In the case of arbitrary parameters α and σ we have obtained the only solitary wave solutions (28) and (29) in the form of kinks.

2. Painlevé analysis and exact solution of equation at $m = 2$ and $n = 1$

Let us consider equation (7) at $m = 2$ and $n = 1$. In this case this the equation takes the form

$$u_t + \alpha\,u\,u_x - \delta\,(u^2\,u_x)_x + u_{xx} + \sigma\,u_{xxx} + u_{xxxx} = 0. \tag{34}$$

Using the traveling wave solutions in the form

$$u(x,t) = w(z), \qquad z = k\,x + \omega\,t, \tag{35}$$

After integrating we have the nonlinear differential equation in the form

$$k^4\,w_{zzz} + k^3\,\sigma\,w_{zz} + k^2\,w_z - \delta\,k^2\,w\,w_z + \frac{\alpha\,k}{2}\,w^2 + \omega\,w + C_1 = 0, \tag{36}$$

where C_1 is a constant of integration.

Let us apply the Painlevé test to study (36). The equation with the leading members in this case is found from equation (36) and takes the form

$$k^4\,w_{zzz} - 6\,k^2\,w\,w_z = 0. \tag{37}$$

The first step of the Painlevé test is the determination of the first member of the expansion for solution in the Laurent series. Substituting

$$w = \frac{a_0}{z^p} \tag{38}$$

into equation with the leading members (37) we have two values $(a_0,\ p) = (k,\ 1)$ and $(a_0,\ p) = (-k,\ 1)$.

The second step of the Painlevé test is the determination of the Fuchs indices of the expansion of the solution. With this aim we substitute

$$w = \pm\frac{k}{z} + b\,z^{j-1} \tag{39}$$

into the equation with the leading members (37) again. As a result we have

$$j = -1, \qquad j = 3, \qquad j = 4. \tag{40}$$

At the third step of the Painlevé analysis of equation (36) we have to check the arbitrary constants at the expansion of the solution in the Laurent series. We substitute

$$w = \pm\frac{k}{z} + a_1 + a_2\,z + a_3\,z^2 + a_4\,z^3 + \dots \tag{41}$$

into equation (36). We obtain the following system of equations:

$$a_1 = \mp \frac{\sigma}{6}, \tag{42}$$

$$a_2 = \mp \frac{\sigma^2}{36\,k} \pm \frac{1}{6\,k} - \frac{\alpha}{12\,k}, \tag{43}$$

$$\omega \mp \frac{\alpha\,k\sigma}{6} = 0, \tag{44}$$

$$C_1 + 4\,\sigma\,k^3 a_3 - \frac{\alpha\,k\sigma^2}{18} + \frac{k\alpha}{4} \mp \frac{k\alpha^2}{8} = 0. \tag{45}$$

Equation (36) would be integrable if equations (44) and (45) are identically equal to zero but as we can see they are not. The equation is equal to zero in the case of

$$\omega = \pm \frac{\alpha\,k\,\sigma}{6} \tag{46}$$

Let us look for the exact solutions of equation (36) using the method of the logistic function. We search the solution in the form

$$w = A_0 + A_1\,Q(z), \tag{47}$$

where $Q(z)$ is the logistic function

$$Q(z) = \frac{1}{1 + e^{-z+z_0}}. \tag{48}$$

Here z_0 is an arbitrary constant.

Taking into account the value ω from (44) we have the following solutions:

$$w_{1,2} = -\frac{\sigma}{6} \mp \frac{m_1}{6}\,(1 - 2\,Q(z)), \quad z = \frac{m_1}{3}\left(\pm x \pm \frac{\alpha\,\sigma}{6}\,t\right), \qquad m_1 = \sqrt{18 - 3\,\sigma^2 - 9\,\alpha}, \tag{49}$$

at the additional condition

$$C_1^{(1)} = \mp \frac{\alpha\,m_1^{1/2}}{216}\left(4\,\sigma^2 - 18 + 9\,\alpha\right). \tag{50}$$

and two other solutions in the form

$$w_{3,4} = \frac{\sigma}{6} \pm \frac{m_2}{6}\,(1 - 2\,Q(z)), \quad z = \frac{m_2}{3}\left(\pm x \mp \frac{\alpha\,\sigma}{6}\,t\right), \qquad m_2 = \sqrt{18 - 3\,\sigma^2 + 9\,\alpha}, \tag{51}$$

at the additional condition

$$C_1^{(2)} = \pm \frac{\alpha\,m_2^{1/2}}{216}\left(4\,\sigma^2 - 18 - 9\,\alpha\right). \tag{52}$$

Exact solutions (49) and (51) are the solitary wave solutions in the form of kinks.

We can look for the periodic solutions of equation (36) if we apply the method developed in works [50–52].

3. Painlevé analysis and exact solutions of the equation at $m = 2$ and $n = 2$

Let us consider equation (10) at $m = 2$ and $n = 2$. In this case the nonlinear differential equation takes the form

$$k^4\, w_{zzz} + k^3\, \sigma\, w_{zz} + k^2\, w_z - 6\, k^2\, w\, w_z + \frac{\alpha}{3}\, w^3 + \omega\, w + C_1 = 0. \tag{53}$$

The general solution of equation (53) has the pole of the first order because substituting $w = \frac{a_0}{z^p}$ we have two values $(a0,\ p) = (k,\ 1)$ and $(a_0,\ p) = (-k,\ 1)$.

The Fuchs indices correspond to the previous case:

$$j_1 = -1, \qquad j_2 = 3, \qquad j_3 = 4. \tag{54}$$

Substituting (41) into equation (53) we obtain the system of equations in the form

$$a_1 = \mp\left(\frac{\alpha}{36}\,\alpha + \frac{\sigma}{6}\right), \tag{55}$$

$$a_2 = \mp\left(\frac{\sigma\,\alpha}{54\,k} + \frac{5\,\alpha^2}{1296\,k} + \frac{1}{6\,k} - \frac{\sigma^2}{36\,k}\right), \tag{56}$$

$$\pm\left(\omega\,k + \frac{\alpha^2 k^2 \sigma}{36} + \frac{\alpha^3 k^2}{216} + \frac{\alpha\,k^2}{6}\right) = 0, \tag{57}$$

$$\begin{gathered} C_1 + \frac{4\,\alpha\,k^3 a_3}{3} + 4\,\sigma\,k^3 a_3 \pm \\ \pm\left(\frac{\alpha\,k\sigma^3}{81} - \frac{\alpha\,k\,\sigma}{18} - \frac{\alpha^3\,k\,\sigma}{486} - \frac{\alpha^2 k \sigma^2}{324} - \frac{7\,\alpha^4 k}{34992} - \frac{\alpha^2\,k}{108}\right) = 0. \end{gathered} \tag{58}$$

One can see that equation (53) does not pass the Painlevé test because equations (57) and (58) are not equal to zero. We notece that there is the expansion of the general solution of equation (53) in the Laurent series, but we have only unique arbitrary constant in this expansion. From equation (57) we can find the dependence ω in the form:

$$\omega = -\frac{\alpha^2 k \sigma}{36} - \frac{\alpha^3 k}{216} - \frac{\alpha\,k}{6}. \tag{59}$$

In this case the coefficient a_3 of solution for equation (53) can be taken as an arbitrary coefficient. From equation (58) one can also find the condition for the constant C_1.

The exact solutions of equation (53) can be found using the formula:

$$w = B_0 + B_1\, Q(z), \tag{60}$$

where B_0 and B_1 are constant coefficients found from the Laurent series.

As a result we have the solution in the form

$$\begin{gathered} w_{1,2} = -\frac{\alpha}{36} - \frac{\sigma}{6} \mp \frac{m_1}{36} \pm \frac{m_1}{18}\, Q(z), \\ z = \pm\frac{m_1}{18} x \mp \frac{m_1\,\alpha}{3888}\left(6\,\sigma\,\alpha + \alpha^2 + 36\right)\, t, \end{gathered} \tag{61}$$

where m_1 takes the form

$$m_1 = \sqrt{15\,\alpha^2 + 72\,\sigma\,\alpha - 108\,\sigma^2 + 648}. \tag{62}$$

Solution (61) satisfies equation (53) at the additional condition

$$C_1^{(1,2)} = \pm\frac{\alpha\,m_1}{629856}\,(\alpha + 6\,\sigma)\left(7\,\alpha^2 + 30\,\sigma\,\alpha - 72\,\sigma^2 + 324\right). \tag{63}$$

The two other solutions of equation (53) can be written as

$$\begin{gathered} w_{3,4} = \frac{\alpha}{36} + \frac{\sigma}{6} \pm \frac{m_2}{36} \mp \frac{m_2}{18}\,Q(z), \\ z = \pm\frac{m_2}{18}x \mp \frac{m_2\,\alpha}{3888}\left(6\,\sigma\,\alpha + \alpha^2 + 36\right)\,t, \end{gathered} \tag{64}$$

where m_2 takes the form

$$m_2 = \sqrt{15\,\alpha^2 + 72\,\sigma\,\alpha - 108\,\sigma^2 + 648}. \tag{65}$$

Solution (61) satisfies equation (53) at additional condition

$$C_1^{(3,4)} = \mp\frac{\alpha\,m_2}{629856}\,(\alpha + 6\,\sigma)\left(7\,\alpha^2 + 30\,\sigma\,\alpha - 72\,\sigma^2 + 324\right). \tag{66}$$

These solutions of equation (53) are the solitary wave solutions in the form of kinks. Let us note that we can obtain the periodic solutions of equation (53) if we apply the method developed in papers [50–52].

4. Painlevé analysis and exact solutions of the equation at $m = 1$ and $n = 3$

Let us study the existence of the exact solution of equation (10) at $m = 1$ and $n = 3$. It takes the form

$$k^4\,w_{zzz} + k^3\,\sigma\,w_{zz} + k^2\,w_z - 6\,k^2\,w\,w_z + 6\,k\,w^4 + \omega\,w + C_1 = 0, \tag{67}$$

Substituting

$$w = \frac{a_0}{z^p} \tag{68}$$

into equation (67) we have $p = 1$ and

$$a_0 = k, \qquad a_{1,2} = -\frac{k}{2} \pm \frac{i\,k\,\sqrt{3}}{2}. \tag{69}$$

The general solution of equation (67) has the pole of the first order. Substituting $w = \frac{a_0}{z} + b\,z^{j-1}$ into the equation with the leading members in the form

$$k^4\,w_{zzz} + 6\,k\,w^4 = 0, \tag{70}$$

we have the following Fuchs indices:

$$j_1 = -1, \qquad j_{2,3} = \frac{7}{2} + \frac{i}{2}\sqrt{23}. \tag{71}$$

Using the expansion of solution in the Laurent series

$$w = \frac{k}{z} + a_1 + a_2\,z + a_3\,z^2 + \dots \tag{72}$$

into equation (67) we obtain the coefficients in the form

$$a_1 = -\frac{\sigma}{12} - \frac{1}{4}, \tag{73}$$

$$a_2 = \frac{1}{96\,k} - \frac{\sigma}{24\,k} - \frac{\sigma^2}{96\,k}, \tag{74}$$

$$a_3 = -\frac{5\,\sigma^2}{384\,k^2} - \frac{\omega}{24\,k^3} - \frac{5\,\sigma}{384\,k^2} - \frac{7\,\sigma^3}{3456\,k^2} + \frac{3}{128\,k^2}, \tag{75}$$

$$\begin{gathered} a_4 = \frac{71}{4608\,k^3} - \frac{C_1}{30\,k^4} - \frac{\omega\,\sigma}{360\,k^4} - \frac{49\,\sigma^4}{207360\,k^3} - \frac{\omega}{40\,k^4} - \\ - \frac{17\,\sigma^2}{2304\,k^3} - \frac{\sigma^3}{384\,k^3} + \frac{7\,\sigma}{5760\,k^3}, \end{gathered} \tag{76}$$

and so on.

We have the expansion of the solution in the Laurent series but there is only one arbitrary constant z_0 in this series. So we have the solution with one arbitrary constant. This solution also can be found using the logistic function.

For example using the expression

$$w = B_0 + B_1\,Q(z) \tag{77}$$

and substituting it into equation (67) we have

$$w_{1,2}(z) = -\frac{m_1}{8} - \frac{1}{4} - \frac{\sigma}{12} + \frac{m_1}{4}\,Q(z), \tag{78}$$

where

$$\begin{gathered} z = \frac{m_1}{4}\,x - \frac{(\sigma+3)\,(7\,\sigma^2 + 24\,\sigma - 27)\,(\sigma^2 - 1 + 4\,\sigma)}{288\,m_1}\,t, \\ m_1 = \sqrt{2 - 8\,\sigma - 2\,\sigma^2}. \end{gathered} \tag{79}$$

Solution (78) satisfies equation (67) at the additional condition for the constant C_1

$$C_1 = \frac{1}{9216}\,\frac{(\sigma^2 - 21)\,(11\,\sigma^2 + 48\,\sigma + 9)\,(-1 + 4\,\sigma + \sigma^2)}{\sqrt{2 - 8\,\sigma - 2\,\sigma^2}}. \tag{80}$$

Other solutions depend on complex parameters.

5. Painlevé analysis and exact solutions of the equation at $m = 2$ and $n = 3$

Now let us consider equation (10) at $m = 2$ and $n = 3$. Using the traveling wave solution, equation (10) at $\alpha = 4$ and $\delta = 5$ takes the form

$$k^4 w_{zzz} + \sigma k^3 w_{zz} + k^2 w_z - 5 k^2 w^2 w_z + k w^4 + \omega w + C_1 = 0. \tag{81}$$

Equation with the leading members can be written as the following

$$k^4 w_{zzz} - 5 k^2 w^2 w_z + k w^4 = 0. \tag{82}$$

Substituting (11) into (82) we obtain that there is three branches of solutions with the pole of the first order and values

$$a_0^{(1)} = k, \qquad a_0^{(2,3)} = (\pm\sqrt{3} - 3)\, k. \tag{83}$$

Let us consider the expansion of the solution with $a_0^{(1)}$. Substituting expression (12) into (82) we have the following Fuchs indices

$$l_1 = -1, \qquad j_{2,3} = \frac{7}{2} \pm \frac{i\sqrt{3}}{2} \tag{84}$$

We have obtained that equation (81) does not pass the Painlevé test. However, there is the expansion of solution in the Laurent series. We have

$$a_1 = -\frac{\sigma}{7}, \tag{85}$$

$$a_2 = \frac{1}{9\,k} - \frac{11\,\sigma^2}{441\,k}, \tag{86}$$

$$a_3 = -\frac{8\,\sigma^3}{1029\,k^2} + \frac{\sigma}{21\,k^2} - \frac{\omega}{4\,k^3}, \tag{87}$$

$$a_4 = -\frac{2}{81\,k^3} - \frac{38\,\sigma^2}{2835\,k^3} + \frac{3083\,\sigma^4}{972405\,k^3} + \frac{4\,\omega\,\sigma}{35\,k^4} - \frac{C_1}{5\,k^4} \tag{88}$$

and so on.

We see that there is only one arbitrary constant in the expansion of solution in the Laurent series. So, we can obtain the exact solution which can be found using the logistic function.

We have the following solutions:

$$w_{1,2}(z) = -\frac{\sigma}{7} \mp \frac{1}{21}\sqrt{147 - 33\,\sigma^2}\,(1 - 2\,Q(z)), \tag{89}$$

where

$$\begin{gathered} z = k\,x + \omega\,t, \qquad k_{1,2} = \pm\frac{2}{21}\sqrt{147 - 33\,\sigma^2}, \\ \omega = \frac{k}{1372}\left(1029\,k^3 + 196\,\sigma\,k^2 + 308\,k\sigma^2 + 16\,\sigma^3 - 1372\,k\right). \end{gathered} \tag{90}$$

The other solutions of equation (81) are more cumbersome so let us not present them in this work.

6. Conclusion

In this paper we have considered the generalized Kuramoto – Sivashinsky equation (7) using the Painlevé analysis for nonlinear ordinary differential equations. We have obtained that the whole class of equations is not integrable but the expansion of solution for these equations in the Laurent series contains one or two arbitrary constants and as a consequence can have exact solutions. Using the logistic function as the new variable we have found exact solutions for all versions of these KurSivEq equations.

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Метод логистической функции для нахождения аналитических решений нелинейных дифференциальных уравнений

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Ключевые слова: логистическая функция, нелинейная волна, нелинейное обыкновенное дифференциальное уравнение, тест Пенлеве, точное решение

Для нахождения точных решений нелинейных дифференциальных уравнений используется метод логистической функции. Применение метода иллюстрируется на примере нелинейного обыкновенного дифференциального уравнения четвертого порядка. Представлены аналитические решения, полученные с помощью этого метода. Как оказалось, эти решения выражаются через экспоненциальные функции.

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