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Solution to a Parabolic Differential Equation in Hilbert Space Via Feynman Formula - I

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A parabolic partial differential equation $u'_t(t,x) = Lu(t,x)$ is considered, where L is a linear second-order differential operator with time-independent coefficients, which may depend on x. We assume that the spatial coordinate x belongs to a finite- or infinite-dimensional real separable Hilbert space H.

Assuming the existence of a strongly continuous resolving semigroup for this equation, we construct a representation of this semigroup by a Feynman formula, i.e. we write it in the form of the limit of a multiple integral over H as the multiplicity of the integral tends to infinity. This representation gives a unique solution to the Cauchy problem in the uniform closure of the set of smooth cylindrical functions on H. Moreover, this solution depends continuously on the initial condition. In the case where the coefficient of the first-derivative term in L vanishes we prove that the strongly continuous resolving semigroup exists (this implies the existence of the unique solution to the Cauchy problem in the class mentioned above) and that the solution to the Cauchy problem depends continuously on the coefficients of the equation.

The article is published in the author's wording.

1. Introduction

Representation of a function by the limit of a multiple integral as multiplicity tends to infinity is called a Feynman formula, after R.P. Feynman, who was the first to use such representations on the physical level of rigor for the solution of the Cauchy problem for PDEs [24, 25]. The term "Feynman formula" in this sense was introduced in 2002 by

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O.G. Smolyanov [31]. One can find out more about the research into Feynman formulas up to 2009 in [33]. The most recent (but not complete) overview is [37] (2014, in Russian). It is important to note that Feynman formulas are closely related to Feynman-Kac formulas [30], however the latter will not be studied in the present article. Usage of Feynman and Feynman-Kac formulas includes exact or numerical evaluation of integrals over Gaussian measures on spaces of high or infinite dimension; some useful approaches to this topic are developed in [6, 8].

Differential equations for functions of an infinite-dimensional argument arise in (quantum) field theory and string theory, theory of stochastic processes and financial mathematics. Evolutionary equations (i.e. PDEs in the form $u_t'(t,x) = \dots$) in infinite-dimensional spaces have been studied since 1960s by O.G. Smolyanov, E.T. Shavgulidze, E. Nelson, A.Yu. Khrennikov, S. Albeverio and others. We will mention just some of the publications, which are most recent and relevant for our study.

In [3] the Schrödinger equation in Hilbert space is studied. The equation includes the terms of second, first and zero order, the coefficient of the second order term is constant. The solution to the Cauchy problem is given by a Feynman-Kac-Ito formula.

In [22] a solution to a heat equation in Hilbert space without the terms of the first and zero order is discussed, the coefficient of the second-derivative term is constant. The solution is given in the form of a convolution with the Gaussian measure (analogous to the finite dimensional equation with constant coefficients), the existence of the resolving semigroup is proved. In [14] the solution to the same equation is given by a Feynman-Kac formula.

In [15] the parabolic equation in finite-dimensional space is studied for the case of variable coefficients. Under the assumption that a strongly continuous resolving semigroup exists for the Cauchy problem, Feynman and Feynman-Kac formulas were proven in [15] for the solution.

In [28], for a class of equations in an infinite-dimensional space, with a variable coefficient at the highest derivative (but without first- and zero-order derivatives' terms), a Feynman formula was obtained and the existence of resolving semigroup was proven.

In spaces over the field of p-adic numbers, Feynman and Feynman-Kac formulas for the solutions of the Cauchy problem for evolutionary equations were given in [11, 12].

In [19, 20], Schrödinger and heat equations in $\mathbb{R}^n$ were studied in the case of time-dependent coefficients, and a Chernoff-type theorem was proven for this case.

In [4, 16] Feynman formulas for perturbed semigroups are obtained.

The present article extends my first results in this area [28] to the case of non-zero coefficients at the first- and zero-order derivatives.

I do not provide the technical proofs for the two key theorems to keep the paper short, but give all the background that is relevant for the proofs. The article may be used as a very short introduction to analysis in Hilbert space and to the applications of C_0-semigroup theory in solving evolutionary PDEs.

2. Notation and definitions

The symbol H stands for the real separable Hilbert space with the scalar product $\langle\cdot,\cdot\rangle$. The self-adjoint, positive, non-degenerate (hence injective), linear operator $A\colon H \to H$

is assumed to be defined everywhere on H. The operator A is assumed to be of trace class, which means that for every orthonormal basis (e_k) in H the sum $\sum_{k=1}^{\infty} \langle Ae_k, e_k\rangle = \mathrm{tr}A$ is finite; this sum is called the trace of A (it is independent of the choice of the basis (e_k)).

The symbol $\mathcal{X}$ below stands for any complex Banach space. The symbol $L_b(\mathcal{X}, \mathcal{X})$ stands for space of all linear bounded operators in $\mathcal{X}$, endowed with the classical operator norm.

Symbol $C(M, N)$ will mean the set of all continuous functions from M to N, where M and N are topological spaces.

A function $f\colon H \to \mathbb{R}$ is called cylindrical [5, 10], if there exist vectors $e_1, \dots, e_n$ from H and function $f^n\colon \mathbb{R}^n \to \mathbb{R}$ such that for every $x \in H$ the equality $f(x) = f^n(\langle x, e_1\rangle, \dots, \langle x, e_n\rangle)$ holds. In other words, the function $f\colon H \to \mathbb{R}$ is cylindrical if there exists an $n-$dimensional subspace $H_n \subset H$ and orthogonal projector $P\colon H \to H_n$ such that $f(x) = f(Px)$ for every $x \in H$. The cylindrical function f can be imagined as a function, which is first defined on H_n and then continued to the entire space H in such a way that $f(x) = f(x_0)$ if $x_0 \in H_n$ and $x \in (x_0 + \ker P)$.

Symbol $D = C^{\infty}_{b,c}(H, \mathbb{R})$ stands for the space of all continuous bounded cylindrical functions $H \to \mathbb{R}$ such that they have Fréchet derivatives [17] of all positive integer orders at every point of H, and their Fréchet derivatives of any positive integer order are bounded and continuous.

If $f\colon H \to \mathbb{R}$ is twice Fréchet differentiable, then $f'(x)$ will stand for the first Fréchet derivative of f at the point x, and $f''(x)$ will denote the second derivative. Riesz-Fréchet representation theorem allows us to assume $f'(x) \in H$ and $f''(x) \in L_b(H, H)$ for every $x \in H$.

Symbol $C_b(H, \mathbb{R})$ stands for the Banach space of all bounded continuous functions $H \to \mathbb{R}$, endowed with a uniform norm $\|f\| = \sup_{x\in H} |f(x)|$. It is regarded as a closed subspace of a complex Banach space $C_b(H, \mathbb{C})$.

Let $X = \overline{C^{\infty}_{b,c}(H, \mathbb{R})}$ be the closure of the space D in $C_b(H, \mathbb{R})$. It is clear, that X with the norm $\|f\| = \sup_{x\in H} |f(x)|$ is a Banach space, as it is a closed linear subspace of the Banach space $C_b(H, \mathbb{R})$. Function f belongs to X if and only if there is a sequence of functions $(f_j) \subset D$ such that $\lim_{j\to\infty} f_j = f$, i.e. $\lim_{j\to\infty} \sup_{x\in H} |f(x) - f_j(x)| = 0$.

Symbol $C_b(H, H)$ stands for a Banach space of all bounded continuous functions $B\colon H \to H$, endowed with the uniform norm $\|B\| = \sup_{x\in H} \|B(x)\|$.

Denote $D_H = \{B\colon H \to H \big| \exists N \in \mathbb{N}, b_k \in H, B_k \in D : B(x) = B_1(x)b_1 + \dots + B_N(x)b_N\}$.

Let X_H be the closure of D_H in $C_b(H, H)$.

If $x \in H$, and $R\colon H \to H$ is linear, trace class, positive, non-degenerate operator, then symbol μ^x_R stands for the Gaussian probabilistic measure [1, 5, 34] on H with expectation x and correlation operator R, i.e. the unique sigma-additive measure on Borel sigma-algebra in H such that the equality $\int_H e^{i\langle z,y\rangle}\mu^x_R(dy) = \exp\left(i\langle z, x\rangle - \frac{1}{2}\langle Rz, z\rangle\right)$ holds for every $z \in H$. To make it shorter, we will write μ_R instead of μ^0_R. See section 3.1. for useful formulas about integration over the Gaussian measure.

If $B\colon H \to H$ is a vector field, and $g\colon H \to \mathbb{R}$ and $C\colon H \to \mathbb{R}$ are real-valued functions, then symbol L defines a differential operator on the space of functions $\varphi\colon H \to \mathbb{R}$

$$(L\varphi)(x) := g(x)\mathrm{tr}A\varphi''(x) + \langle \varphi'(x), AB(x)\rangle + C(x)\varphi(x), \quad x \in H.$$

The pair $(\mathcal{L}, M)$ defines a linear operator $\mathcal{L}$ with the domain M. It will be shown in theorem 4.2 that $L(D) \subset X$ when A, B, g and C have certain properties. So (L, D) is a densely defined (on D) operator $L\colon X \supset D \to X$. Here the earlier defined spaces D and X are endowed with the uniform norm, induced from $C_b(H, \mathbb{R})$. Let $(\overline{L}, D_1)$ be the closure of (L, D) in X. This means that

$$D_1 = \{f \in X \big| \exists (f_j) \subset D : \lim_{j\to\infty} f_j = f, \exists \lim_{j\to\infty} Lf_j\},$$

and, if $f \in D_1$, then, by definition, $\overline{L}f = \lim_{j\to\infty} Lf_j$.

If for every fixed first argument $t > 0$ of the function $u\colon [0, +\infty) \times H \to \mathbb{R}$ we have $[x \longmapsto u(t, x)] \in D_1$, then the expression $\overline{L}u(t, x)$ means the result of applying the operator $\overline{L}$ to the function $x \longmapsto u(t, x)$ with the fixed $t > 0$.

Expression $(S_t)_{t\geq 0}$ defines the one-parameter family of linear operators in the space of functions $\varphi\colon H \to \mathbb{R}$

$$(S_t\varphi)(x) := e^{tC(x) - t\frac{\langle AB(x), B(x)\rangle}{g(x)}} \int_H \varphi(x+y) e^{\left\langle \frac{1}{g(x)} B(x), y\right\rangle} \mu_{2tg(x)A}(dy) \text{ for } t > 0, \text{ and } S_0\varphi := \varphi.$$

Remark 2.1. Further, in theorem 4.1, we will prove that for every $t \geq 0$ and for A, B, g and C having certain properties the following holds i) $S_t(X) \subset X$, ii) operator S_t is bounded, and iii) $\frac{d}{dt} S_t\varphi\big|_{t=0} = L\varphi$ for all $\varphi \in D$. This will allow us to use the Chernoff approximation (theorems 3.1, 3.2) and prove the main result of the present article, theorem 4.4.

3. Helpful facts and techniques

3.1. Integration in Hilbert space

Lemma 3.1. ([5], Chapter II, §2, 3°) If a function $\varphi\colon H \to \mathbb{R}$ is cylindrical and measurable, i.e. $\varphi(x) = \varphi^n(\langle x, e_1\rangle, \ldots, \langle x, e_n\rangle)$ for some $n \in \mathbb{N}$, some measurable function $\varphi^n\colon \mathbb{R}^n \to \mathbb{R}$, and some finite orthonormal family of vectors $e_1, \ldots, e_n$ from space H, then

$$\int_H \varphi(y)\mu_A(dy) = \left(\frac{1}{\sqrt{2\pi}}\right)^n \frac{1}{\sqrt{\det M_Q}} \int_{\mathbb{R}^n} \varphi^n(z) \exp\left(-\frac{1}{2}\left\langle M_Q^{-1} z, z\right\rangle_{\mathbb{R}^n}\right) dz, \quad (1)$$

where $H_n = \mathrm{span}(e_1, \ldots, e_n)$, and $P\colon H \ni h \longmapsto \langle h, e_1\rangle e_1 + \cdots + \langle h, e_n\rangle e_n \in H_n$, $Q = PA$, $Q\colon H_n \to H_n$, and M_Q is the matrix of the operator Q in basis $e_1, \ldots, e_n$ of the space H_n. If $e_1, \ldots, e_n$ is a full set of eigenvectors of the operator Q, and $q_1, \ldots, q_n$ is the corresponding set of eigenvalues, then

$$\int_H \varphi(y)\mu_A(dy) = \left(\frac{1}{\sqrt{2\pi}}\right)^n \frac{1}{\sqrt{\prod_{i=1}^n q_i}} \int_{\mathbb{R}^n} \varphi^n(z_1, \ldots, z_n) \exp\left(-\sum_{i=1}^n \frac{z_i^2}{2q_i}\right) dz_1 \ldots dz_n. \quad (2)$$

Lemma 3.2. (*Explicit form of some integrals over Gaussian measure*)

Let H be a real separable Hilbert space of finite or infinite dimension, $\widetilde{A}\colon H \to H$ be a linear, trace class, symmetric, positive, non-degenerate operator, $\mu_{\widetilde{A}}$ be the centered Gaussian measure on H with the correlation operator $\widetilde{A}$, and $G\colon H \to H$ be a bounded linear operator. Let w and z be non-zero vectors from H.

Then the following equalities hold:

$$\int_H \langle Gy, y\rangle \mu_{\widetilde{A}}(dy) = \mathrm{tr}(\widetilde{A}G), \tag{3}$$

$$\int_H e^{\langle z,y\rangle} \mu_{\widetilde{A}}(dy) = e^{\frac{1}{2}\langle \widetilde{A}z,z\rangle}, \tag{4}$$

$$\int_H \langle w, y\rangle e^{\langle z,y\rangle} \mu_{\widetilde{A}}(dy) = \langle \widetilde{A}w, z\rangle e^{\frac{1}{2}\langle \widetilde{A}z,z\rangle}, \tag{5}$$

$$\int_H \langle Gy, y\rangle e^{\langle z,y\rangle} \mu_{\widetilde{A}}(dy) = (\mathrm{tr}\widetilde{A}G + \langle G\widetilde{A}z, \widetilde{A}z\rangle) e^{\frac{1}{2}\langle \widetilde{A}z,z\rangle}. \tag{6}$$

Proof. Formulas (3) and (4) can be found in [5], chapter II, §2, 1°. Formula (5) can be derived from the fact that the function under the integral is cylindrical, so lemma 3.1 can be employed. For a proof of (6), one can make the change of variable in the integral, $h = y - Aw$, then ([5], chapter II, §4, 2°, theorem 4.2) we have $\mu_{\widetilde{A}}(dy) = e^{-\frac{1}{2}\langle \widetilde{A}w,w\rangle - \langle h,w\rangle} \mu_{\widetilde{A}}(dh)$, and the integral reduces to (3).

Lemma 3.3. (*On a linear change of variable in the integral over Gaussian measure*) Let H be a real separable Hilbert space. Suppose a linear operator $A\colon H \to H$ is positive, non-degenerate, trace class, and self-adjoint. We will identify with the symbol μ_A the centered Gaussian measure on H with the correlational operator A. Let $t > 0$; the symbol tA denotes operator, that takes $x \in H$ to $tAx \in H$. Let $f\colon H \to \mathbb{R}$ be a continuous integrable function.

Then

$$\int_H f(x)\mu_{tA}(dx) = \int_H f(\sqrt{t}x)\mu_A(dx). \tag{7}$$

Proof uses the uniqueness of the Gaussian measure with a given Fourier transform, and the standard theorem of changing variable in the Lebesgue integral.

Lemma 3.4. (*On integrability of a polynomial multiplied by an exponent*) Let H, A, μ_A be as above, $P\colon \mathbb{R} \to \mathbb{R}$ be a polynomial, and $\beta \in \mathbb{R}$.

Then function $H \ni x \longmapsto P(\|x\|)e^{\beta\|x\|} \in \mathbb{R}$ is integrable over μ_A.

Proof is easy to construct by relying on Fernique's theorem [23], which (applied to this case) says that there exists such $\alpha > 0$ that $\int_H e^{\alpha\|y\|^2}\mu_A(dy) < +\infty$.

3.2. Derivatives of cylindrical functions

Proposition 3.1. Let f be a cylindrical real-valued function on H, i.e. there is a number $n \in \mathbb{N}$ and a function $f^n\colon \mathbb{R}^n \to \mathbb{R}$ such that for every $x \in H$ the equality $f(x) = f^n(\langle x, e_1\rangle, \ldots, \langle x, e_n\rangle)$ holds. A set of vectors $e_1, \ldots, e_n$ can be considered orthonormal without loss of generality. Lets complete this set to an orthonormal basis $(e_k)_{k\in\mathbb{N}}$ in H.

Then:

1. Function f is differentiable in the direction h if and only if the function f^n is differentiable in the direction $(\langle h, e_1\rangle, \dots, \langle h, e_n\rangle) \in \mathbb{R}^n$, and

$$f'(x)h = \Big\langle h, \Big(\partial_1 f^n(\langle x, e_1\rangle, \dots, \langle x, e_n\rangle),\ \dots,\ \partial_n f^n(\langle x, e_1\rangle, \dots, \langle x, e_n\rangle),\ 0,\ 0,\ 0, \dots\Big)\Big\rangle,$$

where the symbol $\partial_j f^n$ defines the partial derivative with respect to the j-th argument of the function f^n, and $(\alpha_1, \dots, \alpha_n, 0, 0, 0, \dots) = \alpha_1 e_1 + \dots + \alpha_n e_n$. If the function f has a Fréchet derivative at the point x, then $f'(x)$ is a vector whose first n coordinates yield the gradient of the function f^n, and the other coordinates are zero:

$$f'(x) = \Big(\partial_1 f^n(\langle x, e_1\rangle, \dots, \langle x, e_n\rangle),\ \dots,\ \partial_n f^n(\langle x, e_1\rangle, \dots, \langle x, e_n\rangle),\ 0,\ 0,\ 0, \dots\Big). \qquad (8)$$

2. Function f has a Fréchet derivative in H if and only if the function f^n has a Fréchet derivative in $\mathbb{R}^n$.

3. Let $A\colon H \to H$ be a trace-class operator (i.e. let $\mathrm{tr}A < \infty$). Then

$$\mathrm{tr}Af''(x) = \sum_{s=1}^{n}\sum_{k=1}^{n}\langle Ae_s, e_k\rangle\Big(\partial_k\partial_s f^n(\langle x, e_1\rangle, \dots, \langle x, e_n\rangle)\Big) =$$

$$= \mathrm{tr}\Big(A_n (f^n)''(\langle x, e_1\rangle, \dots, \langle x, e_n\rangle)\Big), \qquad (9)$$

where A_n is the matrix of the operator PA in the basis $e_1, \dots, e_n$, where P is the projector to the linear span of the vectors $e_1, \dots, e_n$.

Proof is a straight-forward application the derivative's definition.

3.3. Differential operator on a finite-dimensional space

Lemma 3.5. ([7], theorems 4.3.1, 4.3.2. and Corollary 4.3.4) Suppose for every $i = 1, \dots, n$ and $j = 1, \dots, n$ functions $a^{ij}\colon \mathbb{R}^n \to \mathbb{R}$, $b^i\colon \mathbb{R}^n \to \mathbb{R}$, $c\colon \mathbb{R}^n \to \mathbb{R}$ from $C_b^\infty(\mathbb{R}^n, \mathbb{R})$ are given, where $C_b^\infty(\mathbb{R}^n, \mathbb{R})$ is the class of all bounded real-valued functions on $\mathbb{R}^n$, which have bounded partial derivatives of all orders. Suppose also that $c(x) \le 0$ for all $x \in \mathbb{R}^n$.

For $u \in C_b^\infty(\mathbb{R}^n, \mathbb{R})$ we define a differential operator T by the formula

$$(Tu)(x) = \sum_{i=1}^{n}\sum_{j=1}^{n} a^{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j}u(x) + \sum_{i=1}^{n} b^i(x)\frac{\partial}{\partial x_i}u(x) + c(x)u(x).$$

Suppose that there exists a constant $\varkappa > 0$ such that for every $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$ and all $x \in \mathbb{R}^n$ the ellipticity condition is fulfilled: $\sum_{i=1}^{n}\sum_{j=1}^{n} a^{ij}(x)\xi_i\xi_j \ge \varkappa\|\xi\|^2$. Take an arbitrary constant $\lambda > 0$ and function $f \in C_b^\infty(\mathbb{R}^n, \mathbb{R})$.

Then:

1. There is a unique function $u \in C_b^\infty(\mathbb{R}^n, \mathbb{R})$, which is a solution of the equation

$$(Tu)(x) - \lambda u(x) = f(x). \qquad (10)$$

2. For every function $v \in C_b^\infty(\mathbb{R}^n, \mathbb{R})$ the following estimate is true

$$\sup_{x\in\mathbb{R}^n} |(Tv)(x) - \lambda v(x)| \geq \lambda \sup_{x\in\mathbb{R}^n} |v(x)|. \tag{11}$$

Note that equation (10) can have unbounded solutions; this does not contradict the lemma.

3.4. Strongly continuous semigroups of operators and evolutionary equations

Let $\mathcal{X}$ be a complex Banach space.

Definition 3.1. By a strongly continuous one-parameter semigroup $(T_s)_{s\geq 0}$ of linear bounded operators in $\mathcal{X}$ we (following [26, 18]) mean the mapping

$$T\colon [0, +\infty) \to L_b(\mathcal{X}, \mathcal{X})$$

of the non-negative half-line into the space of all bounded linear operators on $\mathcal{X}$, which satisfies the following conditions:

1. $\forall \varphi \in \mathcal{X} : T_0\varphi = \varphi$.
2. $\forall t \geq 0, \forall s \geq 0 : T_{t+s} = T_t \circ T_s$.
3. $\forall \varphi \in \mathcal{X}$ function $s \longmapsto T_s\varphi$ is continuous as a mapping $[0, +\infty) \to \mathcal{X}$.

Definition 3.2. By the generator of a strongly continuous one-parameter semigroup $(T_s)_{s\geq 0}$ of linear bounded operators on $\mathcal{X}$ we mean a linear operator $\overline{\mathcal{L}}\colon \mathcal{X} \supset Dom(\overline{\mathcal{L}}) \to \mathcal{X}$ given by the formula

$$\overline{\mathcal{L}}\varphi = \lim_{s\to+0} \frac{T_s\varphi - \varphi}{s}$$

on its domain

$$Dom(\overline{\mathcal{L}}) = \left\{ \varphi \in \mathcal{X} : \exists \lim_{s\to+0} \frac{T_s\varphi - \varphi}{s} \right\},$$

where the limit is understood in the strong sense, i.e. it is defined in terms of the norm in the space $\mathcal{X}$.

The use of the symbol $\overline{\mathcal{L}}$ for the generator is related to the fact that the generator is always a closed operator:

Proposition 3.2. (theorem 1.4 in [26], p. 51) The generator of a strongly continuous semigroup is a closed linear operator with a dense domain. The generator defines its semigroup uniquely.

Proposition 3.3. (lemma 1.1 and definition 1.2. in [26], p. 48-49) The set $Dom(\overline{\mathcal{L}})$ coincides with the set of those $\varphi \in \mathcal{X}$, for which the mapping $s \longmapsto T_s\varphi$ is differentiable with respect to s at every point $s \in [0, +\infty)$.

Definition 3.3. 1. The problem of finding a function $U\colon [0, +\infty) \to \mathcal{X}$ such that

$$\begin{cases} \frac{d}{dt}U(t) = \overline{\mathcal{L}}U(t); \quad t \geq 0, \\ U(0) = U_0, \end{cases} \tag{12}$$

is called the abstract Cauchy problem, associated with the closed linear operator $\overline{\mathcal{L}}\colon \mathcal{X} \supset Dom(\overline{\mathcal{L}}) \to \mathcal{X}$ and a vector $U_0 \in \mathcal{X}$.

2. A function $U\colon [0, +\infty) \to \mathcal{X}$ is called a classic solution to abstract Cauchy problem (12) if, for every $t \geq 0$, the function U has a continuous derivative $U'\colon [0, +\infty) \to \mathcal{X}$, $U(t) \in Dom(\overline{\mathcal{L}})$, and (12) holds.

3. A continuous function $U\colon [0, +\infty) \to \mathcal{X}$ is called a mild solution to abstract Cauchy problem (12) if for every $t \geq 0$ we have $\int_0^t U(s)ds \in Dom(\overline{\mathcal{L}})$ and $U(t) = \overline{\mathcal{L}} \int_0^t U(s)ds + U_0$.

Proposition 3.4. (proposition 6.2 in [26], p. 145) If the operator $(\overline{\mathcal{L}}, Dom(\overline{\mathcal{L}}))$ is a generator of a strongly continuous semigroup $(T_s)_{s\geq 0}$, then:

1. For every $U_0 \in Dom(\overline{\mathcal{L}})$ there is a unique classic solution to abstract Cauchy problem (12), which is given by the formula $U(t) = T(t)U_0$.

2. For every $U_0 \in \mathcal{X}$ there is a unique mild solution to abstract Cauchy problem (12), which is given by the formula $U(t) = T(t)U_0$.

Definition 3.4. Linear operator $\mathcal{L}\colon \mathcal{X} \supset Dom(\mathcal{L}) \to \mathcal{X}$ in Banach space $\mathcal{X}$ is called dissipative if for every $\lambda > 0$ and every $x \in Dom(\mathcal{L})$ the estimate $\|\mathcal{L}x - \lambda x\| \geq \lambda \|x\|$ holds.

Proposition 3.5. (*On the closability of a densely defined dissipative operator*) (proposition 3.14 in [26]) A linear dissipative operator $\mathcal{L} : \mathcal{X} \supset Dom(\mathcal{L}) \to \mathcal{X}$ in the Banach space $\mathcal{X}$ with the domain $Dom(\mathcal{L})$ dense in $\mathcal{X}$ is closable. The closure $\overline{\mathcal{L}} : \mathcal{X} \supset Dom(\overline{\mathcal{L}}) \to X$ is also a dissipative operator.

The main tool for the construction of Feynman formulas for the solutions of the Cauchy problem is Chernoff's theorem. For convenience we decompose its conditions into several blocks and give them separate names, as follows.

Theorem 3.1. (P. R. Chernoff, 1968; see [35] and theorem 10.7.21 in [2]) Let $\mathcal{X}$ be Banach space, and $L_b(\mathcal{X}, \mathcal{X})$ be the space of all linear bounded operators in $\mathcal{X}$ endowed with the operator norm. Let $\overline{\mathcal{L}}\colon \mathcal{X} \supset Dom(\overline{\mathcal{L}}) \to \mathcal{X}$ be a linear operator.

Suppose there is a function F such that:

(E). There exists a strongly continuous semigroup $(e^{t\overline{\mathcal{L}}})_{t\geq 0}$, and its generator is $(\overline{\mathcal{L}}, Dom(\overline{\mathcal{L}}))$.

(CT1). F is defined on $[0, +\infty)$, takes values in $L_b(\mathcal{X}, \mathcal{X})$ and $t \longmapsto F(t)f$ is continuous for every vector $f \in \mathcal{X}$.

(CT2). $F(0) = I$.

(CT3). There exists a dense subspace $\mathcal{D} \subset \mathcal{X}$ such that for every $f \in \mathcal{D}$ there exists a limit $F'(0)f = \lim_{t\to 0}(F(t)f - f)/t = \mathcal{L}f$.

(CT4). The operator $(\mathcal{L}, \mathcal{D})$ has a closure $(\overline{\mathcal{L}}, Dom(\overline{\mathcal{L}}))$.

(N). There exists $\omega \in \mathbb{R}$ such that $\|F(t)\| \leq e^{\omega t}$ for all $t \geq 0$.

Then for every $f \in \mathcal{X}$ we have $(F(t/n))^n f \to e^{t\overline{\mathcal{L}}} f$ as $n \to \infty$, and the limit is uniform with respect to t from every segment $[0, t_0]$ for every fixed $t_0 > 0$.

Definition 3.5. In the present article two mappings F_1 and F_2 are called Chernoff-equivalent if there exists a C_0-semigroup $(e^{t\overline{\mathcal{L}}})_{t\geq 0}$ such that $(F_1(t/n))^n f \to e^{t\overline{\mathcal{L}}} f$, $(F_2(t/n))^n f \to e^{t\overline{\mathcal{L}}} f$ for every $f \in \mathcal{X}$ as $n \to \infty$, and the limit is uniform with respect to t from every segment $[0, t_0]$ for every fixed $t_0 > 0$.

Remark 3.1. There are several slightly different definitions of the Chernoff equivalence, see e.g. [36, 29, 37]. We will just use this one not going into details. The only thing we need from this definition is that if F satisfies all the conditions of Chernoff's theorem, then by Chernoff's theorem the mapping F is Chernoff-equivalent to the mapping $F_1(t) = e^{t\overline{\mathcal{L}}}$, i.e. the limit of $(F(t/n))^n$ as n tends to infinity yields the C_0-semigroup $(e^{t\overline{\mathcal{L}}})_{t\geq 0}$.

Definition 3.6. Let us following [32] call a mapping F Chernoff-tangent to the operator $\mathcal{L}$ if it satisfies the conditions (CT1)-(CT4) of Chernoff's theorem.

Remark 3.2. With these definitions the Chernoff-equivalence of F to $(e^{t\overline{\mathcal{L}}})_{t\geq 0}$ follows from: existence (E) of the C_0-semigroup + Chernoff-tangency (CT) + growth of the norm bound (N).

Theorem 3.2. (Chernoff-type theorem, [26], corollary 5.3 from theorem 5.2) Let $\mathcal{X}$ be a Banach space, and $L_b(\mathcal{X}, \mathcal{X})$ be the space of all linear bounded operators on $\mathcal{X}$ endowed with the operator norm. Suppose there is a function

$$V\colon [0, +\infty) \to L_b(\mathcal{X}, \mathcal{X}),$$

meeting the condition $V_0 = I$, where I is the identity operator. Suppose there are numbers $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\|(V_t)^k\| \leq Me^{k\omega t}$ for every $t \geq 0$ and every $k \in \mathbb{N}$. Suppose the limit

$$\lim_{t\downarrow 0} \frac{V_t\varphi - \varphi}{t} =: \mathcal{L}\varphi$$

exists for every $\varphi \in \mathcal{D} \subset \mathcal{X}$, where $\mathcal{D}$ is a dense subspace of $\mathcal{X}$. Suppose there is a number $\lambda_0 > \omega$ such that $(\lambda_0 I - \mathcal{L})(\mathcal{D})$ is a dense subspace of $\mathcal{X}$.

Then the closure $\overline{\mathcal{L}}$ of the operator $\mathcal{L}$ is a generator of a strongly continuous semigroup of operators $(T_t)_{t\geq 0}$ given by the formula

$$T_t\varphi = \lim_{n\to\infty} \left(V_{\frac{t}{n}}\right)^n \varphi$$

where the limit exists for every $\varphi \in \mathcal{X}$ and is uniform with respect to $t \in [0, t_0]$ for every $t_0 > 0$. Moreover $(T_t)_{t\geq 0}$ satisfies the estimate $\|T_t\| \leq Me^{\omega t}$ for every $t \geq 0$.

Theorem 3.3. (*Approximation of generator implies approximation of semigroup*) (theorem 4.9 in [26])

Let $(e^{\overline{\mathcal{L}_j}t})_{t\geq 0}$ be a sequence of strongly continuous semigroups of operators in a Banach space $\mathcal{X}$ with the generators $(\overline{\mathcal{L}_j}, Dom(\overline{\mathcal{L}_j}))$, which satisfies, for some fixed constants $M \geq 1, w \in \mathbb{R}$, the condition $\left\|e^{\overline{\mathcal{L}_j}t}\right\| \leq Me^{wt}$ for all $t \geq 0$ and every $j \in \mathbb{N}$. Suppose there is a closed linear operator $(\mathcal{L}, Dom(\mathcal{L}))$ on $\mathcal{X}$ with a dense domain $Dom(\mathcal{L})$, such that $\overline{\mathcal{L}}_j x \to Lx$ for every $x \in Dom(\mathcal{L})$. Suppose the image of the operator $(\lambda_0 I - \mathcal{L})$ is dense in $\mathcal{X}$ for some $\lambda_0 > 0$.

Then the semigroups $(e^{\overline{\mathcal{L}_j}t})_{t\geq 0}, j \in \mathbb{N}$ converge strongly (and uniformly in $t \in [0, t_0]$ for every fixed $t_0 > 0$) to a strongly continuous semigroup $(e^{\overline{\mathcal{L}}t})_{t\geq 0}$ with the generator $\overline{\mathcal{L}}$. In other words, for every $x \in \mathcal{X}$ there exists $\lim_{j\to\infty} e^{\overline{\mathcal{L}_j}t}x = e^{\overline{\mathcal{L}}t}x$ uniformly in $t \in [0, t_0]$ for every fixed $t_0 > 0$.

Remark 3.3. Below, the role of $\mathcal{X}$ will be played by space X, a closed real subspace of the complex Banach space $C_b(H, \mathbb{C})$. Because all the operators used in this paper below are real, and (as it will be proven further in theorems 4.1 and 4.2) X is invariant with respect to them, the above theorems about $\mathcal{X}$ are applicable to X.

3.5. Properties of spaces D, X, D_1

Remark 3.4. It directly follows from the definitions of these spaces that

i) $D \subset D_1 \subset X \subset C_b(H,\mathbb{R}) \subset C_b(H,\mathbb{C})$;

ii) D and D_1 are dense in X;

iii) X is a Banach space.

Proposition 3.6. If $f \in D$, then f is uniformly continuous.

Proof. It follows from the definition of the space D that the function $D \ni f\colon H \to \mathbb{R}$ is bounded and its Fréchet derivatives of all orders exist and are bounded. In particular, there exists $\sup_{x\in H} \|f'(x)\| = M < \infty$. For every $x \in H$ and every $y \in H$ one can see the estimate

$$|f(x) - f(y)| \le \|x - y\| \sup_{z\in[x,y]} \|f'(z)\| \le M\|x - y\| \tag{13}$$

which implies the uniform continuity of f.

□

Proposition 3.7. If $\varphi \in X$, then φ is uniformly continuous.

Proof. Take any given $\varepsilon > 0$. Let us find $\delta > 0$ such that $\|x - y\| < \delta$ implies $|\varphi(x) - \varphi(y)| < \varepsilon$.

As $\varphi \in X$, there exists a sequence of functions $(f_j) \subset D$ converging to φ uniformly. Hence, there exists a number j_0 such that (introducing the notation $f_{j_0} = f$) we have

$$\|\varphi - f_{j_0}\| = \|\varphi - f\| = \sup_{x\in H} |\varphi(x) - f(x)| < \frac{\varepsilon}{3}. \tag{14}$$

Moreover, as $f \in D$, proposition 3.6 implies estimate (13) with some $M > 0$.

Let us set $\delta = \frac{\varepsilon}{3M}$ and note that $\|x - y\| < \delta$. Then

$$|\varphi(x) - \varphi(y)| \le |\varphi(x) - f(x)| + |f(x) - f(y)| + |f(y) - \varphi(y)| \overset{(13),(14)}{<} \frac{\varepsilon}{3} + M\frac{\varepsilon}{3M} + \frac{\varepsilon}{3} = \varepsilon.$$

□

Proposition 3.8. Suppose that a sequence of functions $(f_j)_{j=1}^{\infty} \subset X$ converges uniformly to a function $f_0 \in X$. Then the family $(f_j)_{j=0}^{\infty}$ is equicontinuous.

Proof. Suppose $\varepsilon > 0$ is given. Let us find $\delta > 0$ such that $\|x - y\| < \delta$ implies that $|\varphi(x) - \varphi(y)| < \varepsilon$.

By proposition 3.7, function f_j is uniformly continuous for each $j = 0, 1, 2, \ldots$ Thus, for each $j = 0, 1, 2, \ldots$ there exists $\delta_j > 0$ such that $\|x - y\| < \delta_j$ implies

$$|f_j(x) - f_j(y)| < \frac{\varepsilon}{3}. \tag{15}$$

As $f_j \to f_0$ uniformly, there exists j_0 such that for all $j > j_0$

$$\sup_{x\in H} |f_0(x) - f_j(x)| < \frac{\varepsilon}{3}. \tag{16}$$

Let us set $\delta = \min(\delta_0, \delta_1, \ldots, \delta_{j_0})$. Then for $j > j_0$ we have that $\|x - y\| < \delta$ implies

$$|f_j(x) - f_j(y)| \le |f_j(x) - f_0(x)| + |f_0(x) - f_0(y)| + |f_0(y) - f_j(y)| \overset{(15),(16)}{<} \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Now, if $0 \le j \le j_0$, then $\|x - y\| < \delta$ implies estimate (15), which is even stronger.

□

Remark 3.5. A number $a \in \mathbb{R}$ is called a limit at infinity of a funtion $f\colon H \to \mathbb{R}$ if

$$\lim_{R\to+\infty} \sup_{\|x\|\geq R} |f(x) - a| = 0.$$

It is shown in [28] that if H is infinite-dimensional, then a non-constant function that belongs to X cannot have a limit at infinity. For example, the function $x \longmapsto \exp(-\|x\|^2)$ belongs to $C_b(H, \mathbb{R})$ but not to X.

Remark 3.6. Suppose that $\alpha_k\colon \mathbb{R} \to \mathbb{R}$ is a family of infinitely-smooth functions, uniformly bounded with their first and second derivatives:

$$\sup_{p\in\{0,1,2\}} \sup_{k\in\mathbb{N}} \sup_{t\in\mathbb{R}} \left|\frac{d^p\alpha_k(t)}{dt^p}\right| \leq M \equiv \text{const}.$$

For example, $\alpha_k(t) = \sin(d_k(t - t_k))$, where d_k and t_k are constants and $0 < d_k \leq 1$. Suppose numerical series $\sum_{k=1}^{\infty} b_k$ converges absolutely. Let $(e_k)_{k=1}^{\infty}$ be an orthonormal basis in H.

Then function

$$f(x) = \sum_{k=1}^{\infty} b_k\alpha_k(\langle x, e_k\rangle)$$

belongs to the class D_1.

This statement can be easily extended to the case $\alpha_k\colon \mathbb{R}^{n_k} \to \mathbb{R}$.

Remark 3.7. Space D is not separable (it does not have a countable dense subset). In the case of one-dimensional H it can be shown similar to the standard proof of the nonseparability of $C_b(\mathbb{R}, \mathbb{R})$. If $\dim H > 1$, then $\mathbb{R}^1$ can be embedded into H as a linear span of a non-zero vector $e \in H$. Using this, one can embed the set of cylindrical functions contributing to the non-separability of D in the case of one-dimensional H, into the space D in the general case.

Remark 3.8. By Remark 3.7 and the inclusion $D \subset D_1 \subset X$, one can see that D_1 and X are not separable too.

4. Main results

4.1. Family S_t provides a semigroup with generator $\overline{L}$

Theorem 4.1. (*On the properties of family $(S_t)_{t\geq 0}$ and its connection to the operator L*)

Suppose that $g \in X$, and for every $x \in H$ we have $g(x) \geq g_o \equiv \text{const} > 0$. Suppose that $B \in X_H$ and $C \in X$. Suppose that $t > 0$, and $\mu_{2tg(x)A}$ is the centered Gaussian measure on H with the correlation operator $2tg(x)A$.

For $t \geq 0$ and $\varphi \in C_b(H, \mathbb{R})$ let us define

$$(S_t\varphi)(x) := e^{tC(x)-t\frac{\langle AB(x),B(x)\rangle}{g(x)}} \int_H \varphi(x+y)e^{\left\langle \frac{1}{g(x)}B(x),y\right\rangle}\mu_{2tg(x)A}(dy) \text{ for } t > 0, \text{ and } S_0\varphi := \varphi. \tag{17}$$

Then:

1. If $t \geq 0$ and $\varphi \in C_b(H, \mathbb{R})$, then $S_t\varphi \in C_b(H, \mathbb{R})$. For every $t \geq 0$ the operator $S_t \colon C_b(H, \mathbb{R}) \to C_b(H, \mathbb{R})$ is linear and bounded; its norm does not exceed $e^{\left(\frac{2\|A\|\|B\|^2}{g_0} + \|C\|\right)t}$.

2. If $g \in D$, $C \in D$, $B \in D_H$, then the space D for every $t \geq 0$ is invariant with respect to the operator S_t.

3. If $g \in X$, $C \in X$, $B \in X_H$, then the space X for every $t \geq 0$ is invariant with respect to the operator S_t.

4. For every function $\varphi \in D$, for $g \in X$, $C \in X$, $B \in X_H$ there exists (uniformly with respect to $x \in H$) a limit

$$\lim_{t \to 0} \frac{(S_t\varphi)(x) - \varphi(x)}{t} = g(x)\mathrm{tr}A\varphi''(x) + \langle \varphi'(x), AB(x) \rangle + C(x)\varphi(x) = (L\varphi)(x).$$

5. If $\varphi \in X$, $g \in X$, $C \in X$, $B \in X_H$, then the function $[0, +\infty) \ni t \longmapsto S_t\varphi \in X$ is continuous, i.e. if $t_0 \geq 0, t_n \geq 0$ and $t_n \to t_0$, then $\sup_{x \in H} |(S_{t_n}\varphi)(x) - (S_{t_0}\varphi)(x)| \to 0$.

Analogue of theorem 4.1 for finite-dimensional H can be found in [15]. The proof for the case of infinite-dimensional H follows the same general line but is more involved. It uses lemmas from sections 3.1., 3.5. and will be published in a separate paper.

Theorem 4.2. (*On the properties of the operator* L) **Suppose** that for each $x \in H$ the inequalities $g(x) \geq g_o \equiv \mathrm{const} > 0$ and $C(x) \leq 0$ hold. As $C \in X$, there exists a sequence $(C_j) \subset D$ converging to C uniformly; let us additionally require that this sequence can be selected in such a way that $C_j(x) \leq 0$ for all $j \in \mathbb{N}$ and all $x \in H$. The operator $L \colon D \to X$ is defined by the equation

$$(L\varphi)(x) = g(x)\mathrm{tr}A\varphi''(x) + \langle \varphi'(x), AB(x) \rangle + C(x)\varphi(x).$$

Symbol I stands for the identity operator.

Then:

1. If $g \in D$, $C \in D$, $B \in D_H$ and $\varphi \in D$, then $L\varphi \in D$. If $g \in X$, $C \in X$, $B \in X_H$ and $\varphi \in D$, then $L\varphi \in X$.

2. If $g \in D$, $C \in D$, $B \in D_H$, then for each $\lambda > 0$ the operator $\lambda I - L$ is surjective on D, therefore $(\lambda I - L)(D) = D$ is a dense subspace in X.

3. If $g \in D$, $C \in D$, $B \in D_H$, then the operator (L, D) is dissipative and closable.

4. If $g \in X$, $C \in X$, $B = 0$, then for each $\lambda > 0$ the space $(\lambda I - L)(D)$ is dense in X.

5. If $g \in X$, $C \in X$, $B \in X_H$, then the operator (L, D) is dissipative and has the closure $(\overline{L}, D_1)$. The operator $(\overline{L}, D_1)$ is also dissipative.

The proof of theorem 4.2 is based on the results of sections 3.2., 3.3. and will be published in a separate paper. Item 1 follows from the definition of the operator L. Items 2 and 3 are derived from lemma 3.5, proposition 3.1 and proposition 3.5. Item 4 is derived from item 2. Item 5 is obtained by proceeding to the limit in the dissipativity estimate proven in item 3 and then applying proposition 3.5.

Theorem 4.3. (On the connection between the family $(S_t)_{t \geq 0}$ and the semigroup with the generator $\overline{L}$)

Suppose that $g \in X$, $B \in X_H$, $C \in X$, and for every $x \in H$ we have $g(x) \geq g_0 \equiv$ const > 0 and $C(x) \leq 0$. As $C \in X$, there exists a sequence $(C_j) \subset D$, converging to C uniformly; let us additionally claim that this sequence can be selected in such a way that $C_j(x) \leq 0$ for all $j \in \mathbb{N}$ and all $x \in H$. **Then** the following holds:

1. If the closure $(\overline{L}, D_1)$ of the operator (L, D) is a generator of a strongly continuous semigroup $\left(e^{t\overline{L}}\right)_{t\geq 0}$ of linear continuous operators on the space X, then

$$e^{t\overline{L}}\varphi = \lim_{n\to\infty} \left(S_{\frac{t}{n}}\right)^n \varphi, \tag{18}$$

where limit exists for every $\varphi \in X$ and is uniform with respect to $t \in [0, t_0]$ for every $t_0 > 0$.

2. If $B = 0$, then the operator $(\overline{L}, D_1)$ is a generator of a strongly continuous semigroup $\left(e^{\overline{L}t}\right)_{t\geq 0}$ of linear continuous operators on the space X. Moreover for every $t \geq 0$ we have $\left\|e^{\overline{L}t}\right\| \leq 1$, i.e. the semigroup $\left(e^{\overline{L}t}\right)_{t\geq 0}$ is contractive.

3. Suppose $B = 0$, and for all $j \in \mathbb{N}$ the functions $g_j \in X$, $B_j \in X_H$ and $C_j \in X$ are given. Suppose $B_j = 0$ for all $j \in \mathbb{N}$. Suppose there exists a number $\varepsilon_0 > 0$ such that for all $j \in \mathbb{N}$ and all $x \in H$ we have $g_j(x) \geq \varepsilon_0$ and $C_j(x) \leq 0$. Let us denote by the symbol L_j the operator L, which corresponds to the functions g_j, B_j and C_j, and the operator L corresponding to the functions g, B and C will be denoted by L_0. Suppose also that $g_j(x) \to g(x)$ and $C_j(x) \to C(x)$, uniformly with respect to $x \in H$.

Then the (existing by item 2) strongly continuous semigroups $\left(e^{\overline{L_j}t}\right)_{t\geq 0}$ converge strongly (and uniformly with respect to $t \in [0, t_0]$ for every fixed $t_0 > 0$) to the (existing by item 2) strongly continuous semigroup $\left(e^{\overline{L_0}t}\right)_{t\geq 0}$ with the generator $\overline{L_0}$. In other words for every $t_0 > 0$ and every $\varphi \in X$ there exists a limit

$$\lim_{j\to\infty} \left(e^{\overline{L_j}t}\varphi\right)(x) = \left(e^{\overline{L_0}t}\varphi\right)(x), \tag{19}$$

uniformly with respect to $x \in H$ and $t \in [0, t_0]$.

Proof.

1. Recall theorem 3.1 and set $F(t) = S_t$, $\omega = \frac{2\|A\|\|B\|^2}{g_0} + \|C\|$, $\mathcal{X} = X$, $\mathcal{D} = D$, $F'(0) = L$, $G = \overline{L}$. One can see that according to items 1, 4 and 5 of theorem 4.1 and item 5 of theorem 4.2 all the conditions of theorem 3.1 are fulfilled.

2. Note that $C(x) \leq 0$, so $\sup_{x\in H} e^{C(x)} \leq 1$ and for $B = 0$ one obtains the estimate $\|S_t\| \leq 1$. Conditions of theorem 3.2 are fulfilled if one sets $\mathcal{X} = X$, $\mathcal{D} = D$, $\mathcal{L} = L$, $V_t = S_t$, $M = 1$, $\omega = 0$. Indeed, according to item 1 of theorem 4.1, for all $t \geq 0$ the estimate $\|S_t\| \leq e^{\omega t} = 1$ holds true, therefore $\|\left(S_t\right)^k\| \leq 1 \cdot \dots \cdot 1 = 1$. Other conditions of theorem 3.2 follow from item 4 of theorem 4.1 and items 4 and 5 of theorem 4.2.

3. Recall theorem 3.3, and set $\mathcal{X} = X$, $\mathcal{D} = D$, $\mathcal{L} = L_0$, $\mathcal{L}_n = L_j$. One can see that item 2 of this theorem and items 4 and 5 of theorem 4.2 imply all the conditions of theorem 3.3, except for the following one: if $\varphi \in D$, then $\lim\limits_{j\to\infty} L_j\varphi = L_0\varphi$. A simple check shows that this condition is also fulfilled.

$\square$

4.2. Feynman formula solves the Cauchy problem for the parabolic equation

We want to find a function $u\colon [0,+\infty)\times H\to\mathbb{R}$ satisfying the following conditions (we call them Cauchy problem for the parabolic differential equation):

$$\begin{cases} u'_t(t,x)=Lu(x,t); & t\geq 0, x\in H,\\ u(0,x)=u_0(x); & x\in H. \end{cases} \tag{20}$$

To this Cauchy problem, we relate the so-called abstract Cauchy problem (see Definition 3.3), which we define as the following system of conditions upon the function $U\colon [0,+\infty)\to X$:

$$\begin{cases} \frac{d}{dt}U(t)=\overline{L}U(t); & t\geq 0,\\ U(0)=u_0, \end{cases} \tag{21}$$

Remark 4.1. Problem (20) can be considered as problem (21) in the following sense. Function $u\colon (t,x)\longmapsto u(t,x)$ of two variables (t,x) can be considered as a function $u\colon t\longmapsto [x\longmapsto u(t,x)]$ of one variable t, with values in the space of functions of variable x. Then

$$u(t,x)=(U(t))(x),\quad t\geq 0, x\in H.$$

Using this correspondence, we start from Definition 3.3 and define the solution of problem (20).

Definition 4.1. We call a function $u\colon [0,+\infty)\times H\to\mathbb{R}$ *a strong solution* of problem (20) if it satisfies the following conditions:

$$\begin{cases} u(t,\cdot)\in D_1; & t\geq 0,\\ \text{function } t\longmapsto u(t,\cdot) \text{ is continuous;} & t\geq 0,\\ \text{Uniformly for } x\in H\ \exists\lim\limits_{\varepsilon\to 0}\frac{u(t+\varepsilon,x)-u(t,x)}{\varepsilon}=u'_t(t,x); & t\geq 0,\\ u'_t(t,\cdot)\in X; & t\geq 0,\\ \text{Function } t\longmapsto u'_t(t,\cdot) \text{ is continuous;} & t\geq 0,\\ u'_t(t,x)=Lu(x,t); & t\geq 0, x\in H,\\ u(0,x)=u_0(x); & x\in H. \end{cases} \tag{22}$$

Definition 4.2. We call a function $u\colon [0,+\infty)\times H\to\mathbb{R}$ *a mild solution* of problem (20) if it satisfies the following conditions:

$$\begin{cases} u(t,\cdot)\in X; & t\geq 0,\\ \text{Function } t\longmapsto u(t,\cdot) \text{ is continuous;} & t\geq 0,\\ \int_0^t u(s,\cdot)ds\in D_1; & t\geq 0,\\ u(t,x)=L\int_0^t u(s,x)ds+u_0(x); & t\geq 0, x\in H,\\ u_0\in X. \end{cases} \tag{23}$$

Definition 4.3. Let us use the symbol $C([0,+\infty),X)$ for the class of all functions $u\colon [0,+\infty)\times H\to\mathbb{R}$ such that for every $t\geq 0$ the function $x\longmapsto u(t,x)$ belongs to the class X, and the mapping $t\longmapsto u(t,\cdot)\in X$ is continuous for every $t\geq 0$.

Finally, let us state and prove the main result of the article. We use definitions and notation from Section 2.

Theorem 4.4. (On the solution of the Cauchy problem for a parabolic differential equation in Hilbert space)

Suppose $g \in X, C \in X, B \in X_H$. Suppose there is a number $g_0 > 0$ such that for all $x \in H$ we have $g(x) \geq g_0$ and $C(x) \leq 0$. As $C \in X$, there exists a sequence $(C_j) \subset D$, converging to C uniformly; let us additionally require that this sequence can be selected in such a way way that $C_j(x) \leq 0$ for all $j \in \mathbb{N}$ and all $x \in H$.

Then the following holds:

1. If there exists a strongly continuous semigroup with the generator $\overline{L}$, then for every $u_0 \in D_1$ there exists a solution u of problem (22), unique in the class $C([0,+\infty), X)$. The solution depends continuously on u_0, and is given by the formula $u(t,x) = \lim_{n\to\infty} \left(\left(S_{\frac{t}{n}}\right)^n u_0\right)(x)$, where the limit is uniform with respect to $t \in [0, t_0]$ for every $t_0 > 0$.

2. If there exists a strongly continuous semigroup with the generator $\overline{L}$, then for every $u_0 \in X$ there exists a solution u of problem (23), unique in the class $C([0,+\infty), X)$. It depends continuously on u_0, and is given by the formula $u(t,x) = \lim_{n\to\infty} \left(\left(S_{\frac{t}{n}}\right)^n u_0\right)(x)$, where the limit is uniform with respect to $t \in [0, t_0]$ for every $t_0 > 0$.

3. If $B = 0$, then there exists a strongly continuous semigroup with the generator $\overline{L}$. The formula $u(t,x) = \lim_{n\to\infty} \left(\left(S_{\frac{t}{n}}\right)^n u_0\right)(x)$ becomes simpler than in the case $B \neq 0$. Namely, for $B = 0$ we have

$$u(t,x) = \lim_{n\to\infty} \underbrace{\int_H \int_H \cdots \int_H \int_H}_{n} e^{\frac{t}{n}\left(C(x) + \sum_{k=1}^{n-1} C(y_k)\right)} u_0(y_1) \mu^{y_2}_{\frac{2t}{n} g(y_2) A}(dy_1) \mu^{y_3}_{\frac{2t}{n} g(y_3) A}(dy_2) \ldots$$
$$\ldots \mu^{y_n}_{\frac{2t}{n} g(y_n) A}(dy_{n-1}) \mu^{x}_{\frac{2t}{n} g(x) A}(dy_n). \tag{24}$$

In this case the solution u for all $t > 0$ satisfies the estimate $\sup_{x\in H} |u(t,x)| \leq \sup_{x\in H} |u_0(x)|$.

4. Let $B = 0$, and let the functions $g_j \in X$, $B_j \in X_H$ and $C_j \in X$ be given for all $j \in \mathbb{N}$. Let $B_j = 0$ for all $j \in \mathbb{N}$. Suppose there exists $\varepsilon_0 > 0$ such that $g_j(x) \geq \varepsilon_0$ and $C_j(x) \leq 0$ for all $j \in \mathbb{N}$ and all $x \in H$. Let us use the symbol L_j for the operator L that corresponds to the functions g_j, B_j and C_j, and the symbol L_0 for the operator L that corresponds to the functions g, B and C. Suppose also that $g_j(x) \to g(x)$ and $C_j(x) \to C(x)$, uniformly with respect to $x \in H$. We denote as u_j the solution of problems (22) and (23) for the operator L_j. For solution of problems (22) and (23) with the operator L, we use the symbol u.

Then $u_j(t,x)$ converges to $u(t,x)$ as $j \to \infty$, uniformly with respect $x \in H$ and uniformly with respect to $t \in [0, t_0]$ for every fixed $t_0 > 0$.

Remark 4.2. Note that if $B = 0$, then solution depends continuously on the data of the Cauchy problem: the coefficients of the equation (item 4) and the initial condition (items 1 and 2).

Remark 4.3. Analogous theorems for $\mathbb{C}$- or $\mathbb{R}^n$-valued functions u can be formulated mutatis mutandis. The result will hold true due to the theorem above and the linearity of L and S_t. The only additional condition will be that the coefficients of the equation must be real-valued. The same remark is applicable to all the key theorems of this article.

Proof of the theorem.

1. Suppose that there exists a strongly continuous semigroup with the generator $\overline{L}$. Then by item 1 of proposition 3.4 we obtain the existence of a strong solution (definition 3.3) to Cauchy problem (21), and the solution is unique in the class $C([0,+\infty),X)$. By item 1 of theorem 4.3 the semigroup is given in the form described. Using the relation between problems (20) and (21) explained in remark 4.1, we obtain the solution for problem (22). The solution is unique in the class $C([0,+\infty),X)$, as follows from remark 4.1.

2. The proof is similar to that in item 1. The only difference is that in proposition 3.4 we use item 2 instead of item 1.

3. The existence of the sought semigroup follows from item 2 of theorem 4.3. The estimate for the supremum of the absolute value of the solution follows from the fact that the semigroup is contractive.

Let us explain how the equality $u(t,x) = \lim_{n\to\infty}\left(\left(S_{\frac{t}{n}}\right)^n u_0\right)(x)$ implies formula (24). For a continuous bounded function $\psi\colon H\to\mathbb{R}$ and a point $x\in H$, the following change of variables rule in the integral is correct:

$$\int_H \psi(y)\mu_A(dy) = \int_H \psi(y-x)\mu_A^x(dy).$$

Applying this rule, and changing A to $2tg(x)A$, we come to the equality

$$(S_t\varphi)(x) = e^{tC(x)}\int_H \varphi(x+y)\mu_{2tg(x)A}(dy)$$

$$= e^{tC(x)}\int_H \varphi(x+(y-x))\mu^x_{2tg(x)A}(dy) = e^{tC(x)}\int_H \varphi(y)\mu^x_{2tg(x)A}(dy).$$

For $n=2$ in formula (24) we get the expression

$$\left(\left(S_{\frac{t}{2}}\right)^2\varphi\right)(x) = \left(S_{\frac{t}{2}}\left(S_{\frac{t}{2}}\varphi\right)\right)(x) = \int_H\left(\int_H e^{\frac{t}{2}(C(x)+C(y_1))}\varphi(y_1)\mu^{y_2}_{\frac{2t}{2}g(y_2)A}(dy_1)\right)\mu^x_{\frac{2t}{2}g(x)A}(dy_2).$$

In the same way expressions for $n>2$ are derived. Thus, the formula (24) is proven.

4. The proof follows immediately from item 3 of theorem 4.3.

□

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Решение параболического дифференциального уравнения в гильбертовом пространстве с помощью формулы Фейнмана - I

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Ключевые слова: Гильбертово пространство, формула Фейнмана, теорема Чернова, кратные интегралы, гауссовская мера

В работе рассматривается параболическое дифференциальное уравнение $u'_t(t,x) = Lu(t,x)$ в частных производных, где L – это линейный дифференциальный оператор второго порядка с коэффициентами, не зависящими от времени, но зависящими от x. Предполагается, что пространственная переменная x принадлежит конечномерному или бесконечномерному вещественному сепарабельному гильбертову пространству H.

Из существования сильно непрерывной полугруппы, разрешающей рассматриваемое уравнение, в статье выводится представление этой полугруппы в виде формулы Фейнмана, т.е. полугруппа записывается в форме предела кратного интеграла по H при стремящейся к бесконечности кратности. Это представление дает единственное решение задачи Коши для рассматриваемого уравнения в классе функций, являющихся равномерными пределами гладких цилиндрических функций на H. Более того, это решение непрерывно зависит от начального условия. Для случая, когда в операторе L коэффициент при первой производной равен нулю, в настоящей работе доказано, что а) сильно непрерывная разрешающая полугруппа существует (это влечет за собой существование единственного решения для задачи Коши в упомянутом выше классе функций) и б) это решение непрерывно зависит от коэффициентов уравнения.

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