Моделирование и анализ информационных систем. Т. 23, № 1 (2016), с. 23–40
Modeling and Analysis of Information Systems. Vol. 23, No 1 (2016), pp. 23–40

DOI: 10.18255/1818-1015-2016-1-23-40

UDC 519.854.33

A Special Role of Boolean Quadratic Polytopes among Other Combinatorial Polytopes

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Received November 15, 2015

Abstract. We consider several families of combinatorial polytopes associated with the following NP-complete problems: maximum cut, Boolean quadratic programming, quadratic linear ordering, quadratic assignment, set partition, set packing, stable set, 3-assignment. For comparing two families of polytopes we use the following method. We say that a family P is affinely reduced to a family Q if for every polytope $p \in P$ there exists $q \in Q$ such that p is affinely equivalent to q or to a face of q, where $\dim q = O((\dim p)^k)$ for some constant k. Under this comparison the above-mentioned families are splitted into two equivalence classes. We show also that these two classes are simpler (in the above sense) than the families of polytopes of the following problems: set covering, traveling salesman, 0-1 knapsack problem, 3-satisfiability, cubic subgraph, partial ordering. In particular, Boolean quadratic polytopes appear as faces of polytopes in every mentioned families.

Article is published in the author's wording.

Keywords: NP-hard problems, affine reduction, faces of polytopes

For citation: Maksimenko A. N., "A Special Role of Boolean Quadratic Polytopes among Other Combinatorial Polytopes", *Modeling and Analysis of Information Systems*, **23**:1 (2016), 23–40.

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Acknowledgments:
Supported by the project No. 477 of P.G. Demidov Yaroslavl State University within State Assignment for Research.

Introduction

In 1954, Dantzig, Fulkerson, and Johnson [8] solved a 49-city traveling salesman problem via considering a polytope of this problem. This idea turned out quite fruitful. Since then, there were published hundreds of papers about properties of various combinatorial polytopes. In particular, a lot of attention was paid to properties of graphs (1-skeletons) of polytopes (such as criterion of adjacency, diameter, clique number) and complexities of extended formulations.

In this paper we compare combinatorial characteristics of complexity for several families of such polytopes. It is natural to consider the following method for comparing of polytopes. If a polytope p is affinely equivalent to a (not necessary proper) face of a polytope q, then p can not be more complicated than q in any reasonable sense. Below this fact is denoted by $p \le_A q$. If we can compare two polytopes in this sense, then we can compare their characteristics of complexity. For example, if $p \le_A q$ then the graph of p is a subgraph of q, moreover, the face lattice of p is embedded into the face lattice of q.

We note that in recent times the most widespread is a little bit different method for comparing polytopes. This is related with the notion of an extended formulation of a polytope (see for example [7] and [17]). A polytope q is called an *extension* (or an *extended formulation*) of a polytope p if there exists an affine map π with $\pi(q) = p$. The *extension complexity* of a polytope p is the size (i.e. number of facets) of its smallest extension. We will denote by $p \le_E q$ the fact that a (not necessary proper) face of a polytope q is an extension of p. It is clear, that

$$p \le_A q \Rightarrow p \le_E q.$$

For example, it is well known that if p is a convex polytope with n vertices and Δ_n is a simplex with n vertices, then

$$p \le_E \Delta_n. \tag{1}$$

As a rule, the number of vertices of a combinatorial polytope p is exponential in the dimension $\dim p$. Hence, Δ_n in (1) has exponential dimension and the comparison (1) becomes useless in practical sense. So, it is natural to restrict the dimension of q in $p \le_E q$ by some polynomial of $\dim p$. More precisely, let P and Q are sets of polytopes, we will write $P \propto_E Q$ if there exists $k \in \mathbb{N}$ such that for every polytope $p \in P$ there is $q \in Q$ with $p \le_E q$ and $\dim q = O\left((\dim p)^k\right)$. For example, Yannakakis [32] showed that the matching polytopes and the vertex packing polytopes are not more complicated (in the sense of $\propto_E$) than the traveling salesman polytopes. In [20] it was shown that polytopes of any linear combinatorial optimization problem[1] (among them are cut polytopes, 0-1 knapsack polytopes, 3-satisfiability polytopes and many other combinatorial polytopes) are not more complicated than the traveling salesman polytopes. One year later in [22] it was shown that polytopes of any linear combinatorial optimization problem are not more complicated than the cut polytopes and the 0-1 knapsack polytopes. In general, it seems that this statement is true for the family of polytopes of any known NP-hard problem. (The results of this paper are another confirmation.) That is, families of polytopes of NP-hard problems are not distinguishable while comparing by $\le_E$. On the one hand, this is convenient for obtaining the results of a general nature. For example, since the extension complexity of cut polytopes is superpolynomial [13], then the same is true for (almost) all other families of combinatorial polytopes. On the other hand, it is well known that these families are significantly different from each other. For example, any two vertices of the cut polytope constitute an edge (1-face) of this polytope (i.e. the graph of this polytope is complete) [27]. Whereas the checking of nonadjacency of vertices of traveling salesman polytopes is NP-complete [28]. So, it is useful to have a more sensitive method of comparing, like the mentioned above $\le_A$.

With replacing $\le_E$ by $\le_A$ in the definition of $\propto_E$ we will get the definition of *affine reduction* $\propto_A$. Below we will consider only 0/1-polytopes[2]. So, we would like to start

[1] We are talking about the problem of the following form. Assume that in a given set E, each element $e \in E$ has some weight $c(e) \in R$, and $f : 2^E \to \{0,1\}$ is a polynomially (with respect to $|E|$) computable rule. Let $S = \{s \subseteq E \mid f(s) = 1\}$ be the set of all feasible solutions of the problem. We seek for a subset $s \in S$ with the maximal (minimal) summary weight of elements.

with the following example. It has been shown by Billera and Sarangarajan [3] that every 0/1-polytope $p \subset \mathbb{R}^d$ with k vertices is affinely equivalent to a face of the asymmetric traveling salesman polytope:

$$p \leq_A \mathrm{ATSP}_n \quad \text{for } n \geq (4(2^d - k) + 1)d.$$

Note, that n is exponential in d. Hence, this does note imply the affine reduction of 0/1-polytopes to asymmetric traveling salesman polytopes. Moreover, the family of 0/1-polytopes can not be affinely reduced to the family $\{\mathrm{ATSP}_n\}$ for the following reason (see [3, p. 12]). There are at least $2^{2^{d-2}}$ combinatorially non-equivalent d-dimensional 0/1-polytopes for $d \geq 6$ [34, Proposition 8]. On the other hand, if f is the total number of faces of ATSP_n, then

$$f \leq n^2 (n!)^{n^2} \leq n^{n^3+2} = 2^{(n^3+2)\log_2 n}.$$

Therefore, if every d-dimensional 0/1-polytope is a face of ATSP_n, then n is superpolynomial in d.

In [21, 24] it was shown that so-called double covering polytopes are affinely reduced to knapsack polytopes, set covering polytopes, cubic subgraph polytopes, 3-SAT polytopes, partial order polytopes, and traveling salesman polytopes. The linear optimization on double covering polytopes is NP-hard and the problem of checking nonadjacency on these polytopes is NP-complete [25]. Consequently, the same is true for the mentioned families.

In this paper we show, that (in the sense of relation $\propto_A$) Boolean quadratic polytopes (and cut polytopes), quadratic linear ordering polytopes, and quadratic assignment polytopes lie in one equivalence class. Set partition polytopes, set packing polytopes, stable set polytopes, and 3-assignment polytopes lie in another (more complicated) equivalence class and they are simpler (w.r.t. $\propto_A$) than double covering polytopes. Thus, the family of Boolean quadratic polytopes is more pure example of a family of polytopes associated with NP-hard problems. They do not have extra details like NP-completness of adjacency relation and the like. Naturally, the following question arises. Whether this family is "the purest" one? More precisely, is there some family of polytopes P associated with NP-hard problem such that $P \propto_A \mathrm{BQP}$, but $\mathrm{BQP} \not\propto_A P$? The answer to this question was provided in [23]. One can construct infinitely many families of Boolean p-power polytopes (Boolean quadratic polytopes is called Boolean 2-power polytopes) such that each $(p+1)$-family is more pure than p-family for $p \in \mathbb{N}$, $p \geq 2$.

The rest of the paper is organized as follows. Section 1 provides a definition of affine reducibility and its properties. As an example we show that Boolean quadratic polytopes (BQP) are affinely reduced to stable set polytopes (SSP), but SSP can not be affinely reduced to BQP. In section 2 it is shown that SSP are equivalent (in the sense of affine reduction) to set packing polytopes and set partition polytopes. In section 3 we consider double covering polytopes (DCP) and prove that SSP $\propto_A$ DCP, but DCP $\not\propto_A$ SSP. In section 4 it is shown that SSP are equivalent to 3-assignment polytopes. In section 5 we consider quadratic linear ordering polytopes and quadratic assignment polytopes and show that they are equivalent to BQP.

[2] 0/1-polytope is the convex hull of a subset of the vertices of the cube $[0, 1]^d$.

1. Affine reducibility and Boolean quadratic polytopes

Let's consider the following partial order on the set of all convex polytopes.

Definition 1. The fact that a polytope p is affinely equivalent to a polytope q or to a face of q will be denoted by $p \le_A q$. If p is affinely equivalent to q itself, we will use designation $p =_A q$.

This relation is useful for estimation of combinatorial characteristics of polytopes. For example, if $p \le_A q$ then the number of i-faces of p is not greater than the number of i-faces of q for $0 \le i \le \dim p$. It is clear also that the number of facets of p is not greater than the number of facets of q. Furthermore, the graph (1-skeleton) of p is a subgraph of the graph of q. Hence we can compare clique numbers of these graphs and the like. In the case $p \le_A q$ we can also compare the extension complexities of these polytopes [13]. The same is true for the rectangle covering bound [13].

On the other hand, this relation is useless, for example, for estimating diameters of graphs of polytopes. But the diameter of graph can hardly be seen as a characteristic of complexity. It is not greater than 2 for TSP polytopes [26] and it is equal to 1 for Boolean quadratic polytopes [27]. This does not correspond to the real complexity of these problems.

It turns out that this relation allows to form up the currently known families of combinatorial polytopes in hierarchical order. At the very bottom of this hierarchy there are Boolean quadratic polytopes and cut polytopes.

Boolean quadratic polytope is the convex hull of the set

$$\mathrm{BQP}_n = \left\{ x = (x_{ij}) \in \{0,1\}^{\frac{n(n+1)}{2}} \mid x_{ij} = x_{ii}x_{jj},\ 1 \le i \le j \le n \right\}. \qquad (2)$$

It should be noted that the notation BQP_n is commonly used for the convex hull, but for the sake of brevity we do not make distinctions between polytopes themselves and the sets of its vertices. The same remark applies to all other polytopes discussed below.

This polytope is also known as a correlation polytope [10]. Moreover, BQP_n is directly related by so-called covariant mapping with cut polytope, usually denoted by CUT_n [9]. Using the notation of definition 1 this relationship may be written as

$$\mathrm{BQP}_n =_A \mathrm{CUT}_{n+1}.$$

Let us note that BQP_n is a face of BQP_{n+1} defined by $x_{n+1,n+1} = 0$.

Property 2. $\mathrm{BQP}_n \le_A \mathrm{BQP}_{n+1}$, $n \in \mathbb{N}$.

Note also that such relations are normal for families of combinatorial polytopes.

In order to illustrate the basic ideas of this paper, we consider another family of polytopes, which is closely related with BQP_n.

Throughout the paper, we assume that

$$[k] := \{1, 2, \dots, k\}, \quad k \in \mathbb{N}.$$

Let $G = (V, E)$ is undirected graph with the set of vertices $V = \{v_1, v_2, \dots, v_k\}$ and the set of edges E. To each vertex v_i, $i \in [k]$, we associate the component y_i of the vector $y = (y_1, \dots, y_k) \in \mathbb{R}^k$. The *stable set polytope* of a graph G is the convex hull of the set

$$\mathrm{SSP}_k = \left\{ y \in \{0,1\}^k \mid y_i + y_j \le 1 \text{ for every edge } \{v_i, v_j\} \in E \right\}. \tag{3}$$

This polytope also known as a *vertex packing polytope*. Furthermore, by affine mapping $z_i = 1 - y_i$, $i \in [k]$, it is related to the *vertex covering polytope* of a graph G:

$$\mathrm{VCP}_k = \left\{ z \in \{0,1\}^k \mid z_i + z_j \ge 1 \text{ for } \{v_i, v_j\} \in E \right\}.$$

I.e., $\mathrm{SSP}_k =_A \mathrm{VCP}_k$.

Let us note that BQP_n is uniquely determined for a fixed n, whereas the notation SSP_k hides a set of k-dimensional polytopes. For example, if a graph G has no edges, then SSP_k is a cube. If G is a complete graph, then SSP_k is a simplex. Hereinafter the SSP_k will be associated with "the most complicated" polytope in this set. More precisely, for a polytope p and for fixed k inequality

$$p \le_A \mathrm{SSP}_k,$$

means that there exists $q \in \mathrm{SSP}_k$ such that $p \le_A q$.

Generalizing this agreement, we obtain

Definition 3. Let P and Q are sets of polytopes. Then the record

$$P \le_A Q$$

indicates that for every $p \in P$ there exists $q \in Q$ such that $p \le_A q$.

This agreement allows us to deduce an analogue of the property 2 for SSP_k.

Property 4. $\mathrm{SSP}_k \le_A \mathrm{SSP}_{k+1}$.

Furthermore, using this notation BQP_n and SSP_k can be compared.

Theorem 5. $\mathrm{BQP}_n \le_A \mathrm{SSP}_k$ *for* $k = n(n+1)$.

(A similar result is given in [13], but they used relation $\le_E$ and $k = 2n^2$.)

Proof. The equality $x_{ij} = x_{ii}x_{jj}$ in (2) is equivalent to inequalities

$$\begin{aligned} x_{ii} - x_{ij} &\ge 0, \\ x_{jj} - x_{ij} &\ge 0, \\ x_{ii} + x_{jj} - x_{ij} &\le 1, \end{aligned} \tag{4}$$

subject to $x_{ij} \in \{0,1\}$. It remains to transform each of them in an inequality of the form $y_l + y_m \le 1$ from (3). For this we introduce $n(n+1)$ new variables:

$$\begin{aligned} s_{ij} &= x_{ij}, && 1 \le i < j \le n, \\ t_{ij} &= x_{ii} - x_{ij}, && 1 \le i < j \le n, \\ u_i &= x_{ii}, && 1 \le i \le n, \\ \bar{u}_i &= 1 - x_{ii}, && 1 \le i \le n. \end{aligned} \tag{5}$$

Then the restrictions (4) are equivalent to

$$\begin{aligned} s_{ij} + \bar{u}_j &\le 1, \\ t_{ij} + u_j &\le 1, \\ u_i + \bar{u}_i &= 1, \\ s_{ij} + t_{ij} + \bar{u}_i &= 1, \end{aligned}$$

subject to integrality of all variables. Obviously, the last two equalities (more precisely, $n(n+1)/2$ equalities) define some face of a polytope SSP_k, where $k = n(n+1)$, defined by the system of $n(2n-1)$ inequalities

$$\begin{aligned} s_{ij} + \bar{u}_j &\le 1, \\ t_{ij} + u_j &\le 1, \\ u_i + \bar{u}_i &\le 1, \\ s_{ij} + \bar{u}_i &\le 1, \\ t_{ij} + \bar{u}_i &\le 1. \end{aligned}$$

Moreover, the equalities (5) connect this face with the polytope BQP_n by nondegenerate affine mapping. □

Remark 6. For $k \ge 2$ relation $\mathrm{SSP}_k \le_A \mathrm{BQP}_n$ is not satisfied for any n. Since BQP_n is a 2-neighborly polytope [27].

Relying on definitions 1 and 3, we can introduce an analogue of Cook–Karp–Levin polynomial reducibility [14] for families of polytopes (as it was done in [21]).

Definition 7. A family of polytopes P is *affinely reduced* to a family Q if there exists $r \in \mathbb{N}$ such that for every polytope $p \in P$ there is $q \in Q$ with $p \le_A q$ and $\dim q = O\left((\dim p)^r\right)$. Designation: $P \propto_A Q$.

In such terminology the theorem 5 and the remark 6 take the following form:

$$\mathrm{BQP} \propto_A \mathrm{SSP}, \quad \mathrm{SSP} \not\propto_A \mathrm{BQP},$$

where $\mathrm{BQP} = \{\mathrm{BQP}_n\}$, $\mathrm{SSP} = \{\mathrm{SSP}_k\}$.

We list some obvious properties of this kind of reduction.

Theorem 8. *Let $P \propto_A Q$. Suppose that there are polytopes in P with some of the following properties:*

1) superpolynomial (in the dimension of a polytope) number of vertices and facets,
2) superpolynomial clique number of the graph of a polytope,
3) NP-completeness of nonadjacency relation,
4) superpolynomial extension complexity,
5) superpolynomial rectangle covering bound.

Then there are polytopes in Q with the same properties.

2. Set packing and set partition polytopes

Let $G = \{g_1, \dots, g_n\}$ be a finite set and $S = \{S_1, \dots, S_d\} \subseteq 2^G$ be a set of subsets of G. Consider a subset $T \subseteq S$. If every $g_i \in G$ belongs to no more (no less) than one of elements of T then T is called a *packing (covering) of the set* G. Covering, which is both the packing, called a *partition of the set* G.

Let $A = (a_{ij})$ be $n \times d$ matrix of incidences of elements of G and elements of S:

$$a_{ij} = \begin{cases} 1, & \text{for } g_i \in S_j, \\ 0, & \text{otherwise.} \end{cases}$$

For every subset $T \subseteq S$ we consider its characteristic vector $x = (x_j) \in \mathbb{R}^d$:

$$x_j = \begin{cases} 1, & \text{if } S_j \in T, \\ 0, & \text{otherwise.} \end{cases}$$

Denote the set of all such characteristic vectors by $\mathrm{PACK}_d = \mathrm{PACK}(S)$. It is evident that

$$\mathrm{PACK}_d = \{x \in \{0,1\}^d \mid Ax \le \mathbf{1}\},$$

where $\mathbf{1}$ is the n-dimensional all 1 vector. The convex hull of PACK_d is called the *set packing polytope*.

Partition polytopes are defined similarly. The set of vertices PART_d of the *set partition polytope* satisfies the equality

$$Ax = \mathbf{1}. \tag{6}$$

It is clear that a set partition polytope is a face of a set packing polytope:

$$\mathrm{PART}_d \le_A \mathrm{PACK}_d. \tag{7}$$

Note that a stable set polytope is a special case of a set packing polytope:

$$\mathrm{SSP}_k \le_A \mathrm{PACK}_d \quad \text{for } d = k.$$

It is not difficult to prove that the families $\mathrm{SSP} = \{\mathrm{SSP}_k\}$, $\mathrm{PACK} = \{\mathrm{PACK}_d\}$ and $\mathrm{PART} = \{\mathrm{PART}_d\}$ are equivalent in terms of affine reducibility.

Theorem 9. $\mathrm{SSP} \propto_A \mathrm{PART} \propto_A \mathrm{PACK} \propto_A \mathrm{SSP}$.

Proof. Show that PACK_d is a special case of SSP_k for $k = d$. It is sufficient to note that inequality

$$x_1 + x_2 + \ldots + x_k \le 1$$

is equivalent to the set of inequalities

$$x_i + x_j \le 1, \quad 1 \le i < j \le k,$$

provided $x_i \in \{0,1\}$, $1 \le i \le k$.

The reduction $\mathrm{PART} \propto_A \mathrm{PACK}$ is evident (see (7)).

Now we show that SSP $\propto_A$ PART. Consider auxiliary variables $u_{ij} = 1 - y_i - y_j$, $u_{ij} \in \{0, 1\}$. Then the inequalities $y_i + y_j \leq 1$ in (3) can be replaced by equalities

$$y_i + y_j + u_{ij} = 1.$$

Consequently,

$$\text{SSP}_k \leq_A \text{PART}_d \quad \text{for } d = k + |E| \leq k(k+1)/2.$$

Here E is the set of edges in (3). □

3. Double covering polytopes

The name "double covering polytopes" was used in [21] for a family of polytopes considered in [25].

Definition 10. The *double covering polytope* is the convex hull of the set

$$\text{DCP}_d = \{x \in \{0,1\}^d \mid Bx = \mathbf{2}\},$$

where $B \in \mathbb{R}^{m \times d}$ is a 0-1 matrix, $\mathbf{2}$ is the m-dimensional all 2 vector, and each row of B contains exactly four 1's.

Previously in [21, 24], there have been found the following relations for several families of combinatorial polytopes with the property of NP-completeness of nonadjacency relation.

Theorem 11 ([21, 24])**.** *The family of double covering polytopes is affinely reduced to families of polytopes associated with the following problems: travelling salesman, knapsack, set covering, 3-satisfiability, cubic subgraph, partial ordering.*

Now we prove that stable set polytopes are simpler than double covering polytopes.

Theorem 12. $\text{SSP}_k \leq_A \text{DCP}_d$ *for* $d = k + |E| + 1$*, where* $|E|$ *is the number of edges (inequalities) in the equation* (3)*.*

Proof. Let us look at the equation (3). For every edge $\{v_i, v_j\} \in E$ we consider auxiliary variable $u_{ij} = 1 - y_i - y_j$, $u_{ij} \in \{0, 1\}$. Thus every inequality $y_i + y_j \leq 1$ in (3) can be replaced by equality

$$y_i + y_j + u_{ij} = 1. \tag{8}$$

Let u_0 be yet another auxiliary variable and let $u_0 = 1$. Hence the equality (8) is equivalent to

$$y_i + y_j + u_{ij} + u_0 = 2.$$

According to the definition 10, a system of such equalities together with the requirement of integer variables defines the vertices of a double covering polytope DCP_d for $d = k + |E| + 1$. The constraint $u_0 = 1$ defines a face of this polytope. Moreover, this face is affinely equivalent to the given SSP_k. □

We now show that the affine reducibility in the opposite direction is not possible. Note that the NP-completeness of adjacency relation is inherited by affine reduction (theorem 8). The family of double covering polytopes has this property [25], whereas for a stable set polytope the checking of adjacency is polynomial [6]. Hence, if $P \neq NP$ then DCP can not be affinely reduced to SSP. It turns out that the latter is true even without the assumption $P \neq NP$.

Theorem 13. *For the double covering polytope*

$$P = \operatorname{conv}\{x \in \{0,1\}^4 \mid x_1 + x_2 + x_3 + x_4 = 2\}$$

the relation $P \leq \mathrm{SSP}_k$ *does not hold for any* SSP_k.

Proof. The polytope P has 6 vertices

$$\begin{aligned} x^1 &= (1,1,0,0), \\ x^2 &= (0,0,1,1), \\ x^3 &= (1,0,1,0), \\ x^4 &= (0,1,0,1), \\ x^5 &= (1,0,0,1), \\ x^6 &= (0,1,1,0). \end{aligned}$$

They are splitted into three pairs with the following property

$$x^1 + x^2 = x^3 + x^4 = x^5 + x^6. \tag{9}$$

Assume that P is affine equivalent to some face $\operatorname{conv}\{y^1, \dots, y^6\}$ of SSP_k. It is clear that the vertices $y^1, y^2, \dots, y^6$ inherit the property (9):

$$y^1 + y^2 = y^3 + y^4 = y^5 + y^6. \tag{10}$$

We now show that there are two more vertices y^7 and y^8 of SSP_k with

$$y^7 + y^8 = y^1 + y^2. \tag{11}$$

This means that the intersection of $\operatorname{conv}\{y^7, y^8\}$ and $\operatorname{conv}\{y^1, \dots, y^6\}$ is not empty. Hence $\operatorname{conv}\{y^1, \dots, y^6\}$ is not a face of SSP_k.

For the vertices $y^1 = (y^1_1, \dots, y^1_k)$ and $y^2 = (y^2_1, \dots, y^2_k)$ we consider the set

$$I = \{i \in [k] \mid y^1_i = y^2_i\}.$$

Note that every vertex of SSP_k is a 0-1 vector. Thus, by (10) and (11),

$$y^8_i = y^7_i = \cdots = y^2_i = y^1_i \quad \text{for } i \in I. \tag{12}$$

Therefore, we shall consider only those coordinates which values are different for every pair of vertices:

$$J = \{j \in [k] \mid y^1_j + y^2_j = 1\} = [k] \setminus I.$$

Consider the six sets

$$\begin{aligned} U &= \{j \in J \mid y^1_j = 1\}, & \bar{U} &= \{j \in J \mid y^2_j = 1\} = J \setminus U, \\ V &= \{j \in J \mid y^3_j = 1\}, & \bar{V} &= \{j \in J \mid y^4_j = 1\} = J \setminus V, \\ W &= \{j \in J \mid y^5_j = 1\}, & \bar{W} &= \{j \in J \mid y^6_j = 1\} = J \setminus W. \end{aligned}$$

All six sets are distinct, otherwise there would be identical vertices among y^1, y^2, …, y^6. Under this condition, the two sets

$$\begin{aligned} S &= (U \cap V \cap W) \cup (U \cap \bar{V} \cap \bar{W}) \cup (\bar{U} \cap V \cap \bar{W}) \cup (\bar{U} \cap \bar{V} \cap W), \\ \bar{S} &= (\bar{U} \cap V \cap W) \cup (\bar{U} \cap \bar{V} \cap \bar{W}) \cup (U \cap V \cap \bar{W}) \cup (U \cap \bar{V} \cap W) = J \setminus S \end{aligned}$$

differ from each of the above six.

Now we can define the points y^7 and y^8:

$$\begin{aligned} y^7_i &= y^8_i = y^1_i, \quad i \in I, \\ y^7_i &= 1 - y^8_i = 1, \quad i \in S, \\ y^7_i &= 1 - y^8_i = 0, \quad i \in \bar{S}. \end{aligned}$$

This 0-1 points differ from $y^1, y^2, \ldots, y^6$ and equality (11) is satisfied for them. It remains to prove that y^7 and y^8 belong to the SSP_k. That is if $y_i + y_j \le 1$ holds for y^1, y^2, y^3, y^4, y^5, y^6 then it holds for y^7 and y^8 also.

By equation (12), this condition is satisfied for $i, j \in I$. This is true for $i \in I$ and $j \in J$ also, since

$$\max(y^1_i + y^1_j, y^2_i + y^2_j) = y^1_i + 1 = y^7_i + 1 = \max(y^7_i + y^7_j, y^8_i + y^8_j).$$

It remains to check the condition for $i, j \in J$. If $i \in S$ and $j \in \bar{S}$ then $y^7_i + y^7_j = y^8_i + y^8_j = 1$ and the condition is fulfilled.

Consider the case $i, j \in S$. If $i, j \in U$ then $y^1_i = y^1_j = 1$. Hence $y_i + y_j \le 1$ is violated by y^1 for $i, j \in U$. The same is true if i and j both belong to one of the sets $\bar{U}$, V, $\bar{V}$, W, $\bar{W}$. But for any i and j in S the latter is true. For example, if $i \in U \cap V \cap W$ and $j \in \bar{U} \cap \bar{V} \cap W$ then $i, j \in W$ and so on. Hence for any $i, j \in S$ the inequality $y_i + y_j \le 1$ is violated for at least one of the vertices $y^1, \ldots, y^6$.

The same is true for the case $i, j \in \bar{S}$ by symmetry. □

4. Three index assignment polytopes

Consider a ground set S, $|S| = m$. Coordinates of a vector $x \in \mathbb{R}^{S \times S \times S}$ we denote by $x(s, t, u)$, where $s, t, u \in S$. Three index assignment (or 3-dimensional matching) problem can be formulated as the following 0-1 programming problem:

$$\sum_{s \in S} \sum_{t \in S} \sum_{u \in S} c(s, t, u) \cdot x(s, t, u) \to \max,$$

$$\sum_{s\in S}\sum_{t\in S} x(s,t,u) = 1 \quad \forall u \in S, \tag{13}$$

$$\sum_{s\in S}\sum_{u\in S} x(s,t,u) = 1 \quad \forall t \in S, \tag{14}$$

$$\sum_{t\in S}\sum_{u\in S} x(s,t,u) = 1 \quad \forall s \in S, \tag{15}$$

$$x(s,t,u) \in \{0,1\} \quad \forall s,t,u \in S, \tag{16}$$

where $c(s,t,u) \in \mathbb{R}$ is an input vector. By $3\mathrm{AP}_m$ denote the set of all vectors $x \in \mathbb{R}^{S\times S\times S}$ satisfying restrictions (13)–(16). The convex hull of $3\mathrm{AP}_m$ is called the *(axial) three index assignment polytope.*

The first results about this polytope can be found in [11] and [2]. A more recent survey can be found in [29]. In the Russian-language papers there are given the lower bound for the clique number of the graph of $3\mathrm{AP}_m$ [4] and various properties of noninteger vertices of relaxations of this polytope (see for example [19]).

It is obvious that $3\mathrm{AP}_m$ is a special case of PART_d:

$$3\mathrm{AP}_m \le_A \mathrm{PART}_d \quad \text{for } d = m^3. \tag{17}$$

That is the family of three index assignment polytopes is affinely reduced to set partition polytopes: $3\mathrm{AP} \propto_A \mathrm{PART}$.

Using a standard reduction [14] from 3SAT to 3-dimensional matching, Avis and Tiwary [1] showed that 3SAT polytope is a projection of a face of a three index assignment polytope. That is $3SAT \propto_E 3\mathrm{AP}$ in the sense of relation $\le_E$. However, from inequality (17), theorem 9, theorem 13 and $\mathrm{DCP} \propto_A 3SAT$ [21] it follows that the reduction $3SAT \propto_A 3\mathrm{AP}$ is impossible.

Now we show, that $\mathrm{SSP} \propto_A 3\mathrm{AP}$. Therefore, 3AP lies in one equivalence class (in the sense of $\propto_A$) with SSP, PART, and PACK.

For the graph $G(V,E)$ in the equation (3) we denote by

$$W = \{v \in V \mid v \notin e \text{ for every } e \in E\},$$

the set of isolated vertices.

Theorem 14. $\mathrm{SSP}_k \le 3\mathrm{AP}_m$ *for* $m = 3|E| + 2|W|$.

Proof. The ground set S for the $3\mathrm{AP}_m$ will consist of three types of elements:

1. v and $\bar{v}$ for every isolated vertex $v \in W$.

2. e for every edge $e \in E$.

3. (e,v) for every $e \in E$ and $v \in e$.

Now we construct the set of triples $Q \subset S \times S \times S$ such that the face

$$F = \big\{x \in \mathrm{conv}(3\mathrm{AP}_m) \mid x(q) = 0 \ \forall q \notin Q\big\}$$

of $\mathrm{conv}(3\mathrm{AP}_m)$ is affinely equivalent to the $\mathrm{conv}(\mathrm{SSP}_k)$.

For every $v \in W$ the set Q contains four triples: (v, v, v), $(\bar{v}, \bar{v}, \bar{v})$, $(v, v, \bar{v})$, $(\bar{v}, \bar{v}, v)$. There are no other triples containing v or $\bar{v}$. Hence, if $x \in F$ then for every $v \in W$ we have only two cases:

$$x(v, v, v) = x(\bar{v}, \bar{v}, \bar{v}) = 1 \quad \text{or} \quad x(v, v, \bar{v}) = x(\bar{v}, \bar{v}, v) = 1.$$

Now we consider elements e and (e, v) of the set S, where $e \in E$ and $v \in e$. For every nonisolated vertex $v \in V \setminus W$ consider the set of incident edges $E(v) = \{e_{i_1}, \dots, e_{i_p}\}$, where $p = d_G(v)$ is the degree of v. The set of triples Q contains:

1. $\big(e, e, e\big)$ for every $e \in E$.
2. $\big((e, v), (e, v), (e, v)\big)$ for every $e \in E$ and $v \in e$.
3. $\big(e, e, (e, v)\big)$ for every $e \in E$ and $v \in e$.
4. $\big((e_{i_q}, v), (e_{i_{q+1}}, v), e_{i_q}\big)$ for every nonisolated v and for every $e_{i_q} \in E(v)$, where addition $q + 1$ means to be $1 + q \mod p$.

We list some properties of the vertices of the face F.

Note that for every $(e, v) \in S$ the set Q contains exactly two triples with (e, v) in the third entry: $\big((e, v), (e, v), (e, v)\big)$ and $\big(e, e, (e, v)\big)$. Hence, the equation (13) for $u = (e, v)$ is converted into

$$x\big((e, v), (e, v), (e, v)\big) + x\big(e, e, (e, v)\big) = 1.$$

That is, $x\big((e, v), (e, v), (e, v)\big)$ is linearly expressed in $x\big(e, e, (e, v)\big)$.

Note also that for every $e \in S$ the set Q contains exactly three triples with e in the first entry: $\big(e, e, e\big)$, $\big(e, e, (e, v_1)\big)$, and $\big(e, e, (e, v_2)\big)$, where $e = \{v_1, v_2\}$. Hence, the equation (15) for $s = e$ is converted into

$$x\big(e, e, e\big) + x\big(e, e, (e, v_1)\big) + x\big(e, e, (e, v_2)\big) = 1.$$

That is $x\big(e, e, e\big) = 1 - x\big(e, e, (e, v_1)\big) - x\big(e, e, (e, v_2)\big)$ and

$$x\big(e, e, (e, v_1)\big) + x\big(e, e, (e, v_2)\big) \le 1. \tag{18}$$

Reasoning by analogy, we obtain the following equation

$$x\big((e_{i_q}, v), (e_{i_q}, v), (e_{i_q}, v)\big) + x\big((e_{i_q}, v), (e_{i_{q+1}}, v), e_{i_q}\big) = 1$$

for every nonisolated v and for every $e_{i_q} \in E(v)$, where addition $q + 1$ is performed modulo $p = d_G(v)$. Hence,

$$x\big((e_{i_q}, v), (e_{i_{q+1}}, v), e_{i_q}\big) = 1 - x\big((e_{i_q}, v), (e_{i_q}, v), (e_{i_q}, v)\big) = x\big(e_{i_q}, e_{i_q}, (e_{i_q}, v)\big).$$

Moreover, since

$$x\big((e_{i_{q+1}}, v), (e_{i_{q+1}}, v), (e_{i_{q+1}}, v)\big) + x\big((e_{i_q}, v), (e_{i_{q+1}}, v), e_{i_q}\big) = 1,$$

then

$$x\big(e_{i_{q+1}}, e_{i_{q+1}}, (e_{i_{q+1}}, v)\big) = x\big(e_{i_q}, e_{i_q}, (e_{i_q}, v)\big).$$

That is $x\big(e, e, (e, v)\big) = x\big(e', e', (e', v)\big)$ for any two edges e and e', $v \in e$, $v \in e'$.

It is not difficult to see that for the vertices of the face F all variables $x(s, t, u)$ are expressed linearly in terms of $x\big(e, e, (e, v)\big)$ and an inequality (18) is an inequality $y_i + y_j \le 1$ in (3). $\square$

Remark 15. The obtained results can be easily generalized to the case of p index assignment problem ($p > 3$). By analogy, the vertices $p\text{-AP}_m$ of an p index assignment polytope are 0-1 vectors $x \in \mathbb{R}^{m^p}$. The coordinates $x_{i_1 i_2 \dots i_p}$ ($i_1, i_2, \dots, i_p \in \{1, 2, \dots, p\}$) satisfy the following equalities:

$$\sum_{i_2, i_3, \dots, i_p} x_{i_1 i_2 \dots i_p} = 1 \quad \forall i_1 \in \{1, \dots, p\},$$
$$\sum_{i_1, i_3, i_4, \dots, i_p} x_{i_1 i_2 \dots i_p} = 1 \quad \forall i_2 \in \{1, \dots, p\},$$
$$\dots\dots\dots$$
$$\sum_{i_1, i_2, \dots, i_{p-1}} x_{i_1 i_2 \dots i_p} = 1 \quad \forall i_p \in \{1, \dots, p\}.$$

It is evident that
$$p\text{-AP}_m \le \mathrm{PART}_d, \quad \text{where } d = m^p.$$
On the other hand, the equalities

$$x_{i_1 i_2 \dots i_p} = 0 \quad \forall i_p \ne i_{p-1}$$

determine a face of $p\text{-AP}_m$ and this face is affinely equivalent to $(p-1)\text{-AP}_m$. Therefore, by theorem 14
$$\mathrm{SSP}_k \le p\text{-AP}_m \quad \text{for } m = 2k(k-1).$$

5. Quadratic linear ordering polytopes and quadratic assignment polytopes

We begin by describing the linear ordering problem in terms of graph theory.

Let $D = (V, A)$ be a digraph, where $V = \{1, 2, \dots, m\}$ is a vertex set. We assume that D is complete. That is $(i, j) \in A$ and $(j, i) \in A$ for any $i, j \in V$, $i \ne j$. An acyclic tournament[3] in digraph D is called a *linear ordering*.

Consider a characteristic vector $y \in \mathbb{R}^{m(m-1)/2}$ for a linear ordering L in D. The coordinates y_{ij}, $1 \le i < j \le m$, of y are

$$y_{ij} = \begin{cases} 1 & \text{for } (i, j) \in L, \\ 0 & \text{for } (j, i) \in L. \end{cases}$$

Denote by LOP_m the set of characteristic vectors of all linear orderings in D. The convex hull of LOP_m is called the *linear ordering polytope*. LOP_m can also be defined as the set of integer solutions $y \in \{0, 1\}^{m(m-1)/2}$ of the 3-dicycle inequalities (see for example [15]):

$$0 \le y_{ij} + y_{jk} - y_{ik} \le 1, \quad i < j < k. \tag{19}$$

In [5] the quadratic linear ordering polytope is defined as follows. Let

$$I = \big\{(i, j, k, l) \mid i < j,\ k < l, \text{ and } (i, j) \prec (k, l)\big\},$$

[3]Each pair of vertices in a tournament is connected by exactly one arc.

where $(i,j) \prec (k,l)$ means that either $i < k$ or $i = k$ and $j < l$. For every pair of distinct variables y_{ij} and y_{kl} there is introduced a new variable

$$z_{ijkl} = y_{ij}y_{kl}, \quad (i,j,k,l) \in I. \tag{20}$$

Denote by QLOP_m the set of all vectors $z \in \{0,1\}^d$, $d = \binom{m}{2}\left(\binom{m}{2}+1\right)/2$, with coordinates y_{ij} and z_{ijkl} satisfying conditions (19) and (20). The convex hull of QLOP_m is called the *quadratic linear ordering polytope.*

Theorem 16 ([5]). $\mathrm{QLOP}_m \le_A \mathrm{BQP}_n$ *for* $n = \binom{m}{2}$.

Buchheim, Wiegele, and Zheng [5] exploit this result within a branch-and-cut algorithm for solving the quadratic linear ordering problem.

We show that an affine reduction in the opposite direction is also possible.

Theorem 17. $\mathrm{BQP}_n \le_A \mathrm{QLOP}_m$ *for* $m = 2n$.

Proof. The idea of the proof is simple. LOP_m contains an n-dimensional cube as a proper face. In the transformation LOP_m to QLOP_m this cube turns into a Boolean quadratic polytope.

Note that equalities $y_{ij} = 0$ and $y_{ij} = 1$ defines supporting hyperplanes for LOP_m and for QLOP_m. Let

$$J = \{(2i-1, 2i) \mid i \in [n]\}.$$

We set

$$y_{ij} = 0 \quad \text{for all } (i,j) \notin J,\ 1 \le i < j \le m. \tag{21}$$

Only variables y_{ij} are unfixed where i is odd and $j = i+1$. Let us check 3-dicycle inequalities (19). Suppose $i < j < k$, we have two cases:

1. If $(i,j) \notin J$ then $y_{ij} = y_{ik} = 0$. Thus the inequality (19) is transformed into $0 \le y_{jk} \le 1$.

2. If $(i,j) \in J$ then i is odd, $j = i+1$ is even, and $k > i+1$. Hence, the inequality (19) is equivalent to $0 \le y_{ij} \le 1$.

Therefore, n variables $y_{i\,i+1}$, where i is odd, may take the values 0 or 1 independently of each other. Consequently, hyperplanes (21) define a face of LOP_m and this face is an n-dimensional cube.

Look at the variables z_{ijkl}, $(i,j,k,l) \in I$. If $(i,j) \notin J$ or $(k,l) \notin J$, then $z_{ijkl} = 0$. In the case $(i,j) \in J$ and $(k,l) \in J$ we have $z_{ijkl} = y_{ij}y_{kl}$, and besides y_{ij} and y_{kl} are free variables.

Thus there is the following nondegenerate affine map between the face of QLOP_m and BQP_n:

$$x_{ii} = y_{2i-1,\,2i}, \quad 1 \le i \le n,$$
$$x_{ij} = z_{2i-1,\,2i,\,2j-1,\,2j} = y_{2i-1,\,2i} \cdot y_{2j-1,\,2j}, \quad 1 \le i < j \le n.$$

□

The story for quadratic assignment polytopes is repeated almost exactly.

The set of vertices $2\mathrm{AP}_m$ of the *assignment polytope* (or *Birkhoff polytope*) consists of vectors $y \in \{0,1\}^{m\times m}$ satisfying the conditions

$$\sum_j y_{ij} = 1, \ \forall i \in [m], \tag{22}$$

$$\sum_i y_{ij} = 1, \ \forall j \in [m]. \tag{23}$$

Define new variable z_{ijkl} like (20):

$$z_{ijkl} = y_{ij} y_{kl}, \text{ where } (i,j) \prec (k,l). \tag{24}$$

Denote by QAP_m the set of all vectors $z \in \{0,1\}^d$, $d = m^2 + \binom{m^2}{2}$, with coordinates y_{ij} and z_{ijkl} satisfying conditions (22), (23), and (24). The convex hull of QAP_m is called the *quadratic assignment polytope*. It is also useful to consider the *quadratic semi-assignment polytope* [31]. In the definition of its vertex set QSAP_m the condition (23) is omitted.

Theorem 18 ([30, 16, 31])**.** $\mathrm{QAP}_m \le_A \mathrm{QSAP}_m \le_A \mathrm{BQP}_n$ *for* $n = m^2$.

This connection is used in [16] for obtaining valid inequalities for QAP_m. In particular, QAP_m is a 2-neighborly polytope (every two vertices constitute an edge of it), since BQP_n is 2-neighborly. In [16] it is also shown that the linear ordering polytope LOP_m and the traveling salesman polytope TSP_m are projections of QAP_m:

$$\mathrm{LOP}_m \le_E \mathrm{QAP}_m, \quad \mathrm{TSP}_m \le_E \mathrm{QAP}_m.$$

Note that the affine reductions LOP $\propto_A$ QAP and TSP $\propto_A$ QAP are impossible, since LOP_m is not 2-neighborly for $m \ge 3$ [33] and TSP_m is not 2-neighborly for $m \ge 6$ [26].

Theorem 19. $\mathrm{BQP}_n \le_A \mathrm{QAP}_m$ *for* $m = 2n$.

Proof. By analogy with the proof of theorem 17 it is sufficient to show that the Birkhoff polytope $2\mathrm{AP}_m$ has an n-dimensional cube as a face. Let

$$J = \big\{(i,i) \mid i \in [m]\big\} \cup \big\{(2i-1, 2i) \mid i \in [n]\big\} \cup \big\{(2i, 2i-1) \mid i \in [n]\big\}.$$

Then the equalities

$$y_{ij} = 0 \text{ for every } (i,j) \notin J$$

define the required face. $\square$

6. Resume

Boolean quadratic polytopes, cut polytopes, quadratic linear ordering polytopes, and quadratic assignment polytopes are in one class of equivalence within the framework of affine reducibility. A bit more complicated class contains stable set polytopes, set packing polytopes, set partitioning polytopes, and n-index assignment polytopes for $n \ge 3$.

An even more complicated are double covering polytopes, 3-satisfiability polytopes, set covering polytopes, knapsack polytopes, cubic subgraph polytopes, partial ordering polytopes, traveling salesman polytopes. The problem of partitioning of these families into equivalence classes is not solved completely. Nevertheless, Fiorini [12] proved that k-satisfiability polytopes and m-satisfiability polytopes are in different classes for $k \neq m$. Moreover, all of them are simpler than traveling salesman polytopes. From the other hand, the families of so-called Boolean p-power polytopes also are in different classes for distinct values of p [23]. Besides, Boolean p-power polytopes are simpler than Boolean quadratic polytopes (in the sense of affine reduction).

However, if in the definition of affine reducibility (definition 7) we replace the relation $\leq_A$ by relation $\leq_E$ (recall that we write $p \leq_E q$ if a face of a polytope q is an extension of p) then all the mentioned families of polytopes fall into one class of equivalence, since the polytope P of any linear combinatorial optimization problem[4] is an affine image of a face of BQP_n, where n is polynomial in the dimension of P [22].

Thus the affine reduction is a more delicate instrument (versus extending) for comparing the families of combinatorial polytopes. The most complicated (more precisely, the richest in its properties) is a family of traveling salesman polytopes. Families of Boolean p-power polytopes are more simple than any other of the above. More precisely they contain the minimum number of superfluous details (with respect to other families associated with NP-hard problems). Moreover, apparently, combinatorial and geometric properties determining NP-hardness reach the highest concentration precisely in Boolean p-power polytopes.

Proceed to a more precise formulation. Using the above results, it is easy to derive the following relations.

1. $\mathrm{BQP}_n \leq_A \mathrm{SSP}_k \leq_A \mathrm{PACK}_k$ for $k = n(n+1)$.
2. $\mathrm{BQP}_n \leq_A \mathrm{PART}_k$ for $k = 2n^2$.
3. $\mathrm{BQP}_n \leq_A \mathrm{DCP}_k$ for $k = 2n^2 + 1$.
4. $\mathrm{BQP}_n \leq_A \mathrm{3AP}_k \leq_A p\text{-}\mathrm{AP}_k$ for $k = 6n^2 + 3n$ and $p \geq 3$.
5. $\mathrm{BQP}_n \leq_A \mathrm{QLOP}_k$ for $k = 2n$.
6. $\mathrm{BQP}_n \leq_A \mathrm{QAP}_k$ for $k = 2n$.

That is any characteristic of complexity of BQP_n is inherited by the above families of polytopes. For example, in 2012 Fiorini et al. [13] proved that the extension complexity of BQP_n is exponential in n. Later, the lower bound was improved to 1.5^n by Kaibel and Weltge [18]. Hence, the extension complexity of QLOP_k and QAP_k is also exponential in k. The extension complexity of SSP_k, PART_k, DCP_k, and $\mathrm{3AP}_k$ is $2^{\Omega(n^{1/2})}$. The same conclusions can be done for the clique numbers of graphs of the polytopes, since the clique number for BQP_n is 2^n.

[4] See footnote on the page 24

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Максименко А. Н., "Особое место булевых квадратичных многогранников среди других комбинаторных многогранников", *Моделирование и анализ информационных систем*, **23**:1 (2016), 23–40.

DOI: 10.18255/1818-1015-2016-1-23-40

Аннотация. Рассматриваются несколько семейств комбинаторных многогранников, ассоциированных со следующими NP-полными задачами: максимальный разрез, булево квадратичное программирование, квадратичная задача линейного упорядочения, квадратичные назначения, разбиение и упаковка множества, независимое множество, 3-назначения. Для сравнения двух семейств многогранников используется следующий способ. Будем говорить, что семейство P аффинно сводится к семейству Q, если для каждого многогранника $p \in P$ найдется $q \in Q$ такой, что p аффинно эквивалентен q или некоторой грани q, где $\dim q = O((\dim p)^k)$ для некоторой константы k. При таком способе сравнения упомянутые выше семейства разбиваются на два класса эквивалентности. Показано также, что эти два класса проще (в указанном смысле), чем семейства многогранников следующих задач: покрытие множества, коммивояжер, 0/1 рюкзак, 3-выполнимость, кубический подграф, частичное упорядочение. В частности, булевы квадратичные многогранники оказываются гранями многогранников каждого из упомянутых семейств.

Статья публикуется в авторской редакции.

Ключевые слова: NP-трудные задачи, аффинная сводимость, грани многогранников

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Благодарности:
Работа выполнена при поддержке проекта № 477 базовой части гос. задания на НИР ЯрГУ