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FEM-analysis on Layer-adapted Meshes for Turning Point Problems Exhibiting an Interior Layer

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Abstract. We consider singularly perturbed turning point problems whose solutions exhibit an interior layer. Two suitable layer-adapted mesh-types are presented. For both types we give uniform error estimates in the ε-weighted energy norm for finite elements of higher order. Numerical experiments are used to compare the meshes and to confirm the theoretical findings.

Keywords: singular perturbation, turning point, interior layer, layer-adapted meshes, higher order finite elements

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1. Introduction

In this paper we consider singularly perturbed problems of the type

$$\begin{aligned} -\varepsilon u''(x) + a(x)u'(x) + c(x)u(x) = f(x), \qquad & x \in (-1,1), \\ u(-1) = 0, \quad u(1) = 0, \qquad & \end{aligned} \tag{1a}$$

with a small parameter $0 < \varepsilon \ll 1$. We assume that the data a, c, f are sufficiently smooth and satisfy

$$a(x) = -xb(x), \qquad b(x) > 0, \qquad c(x) \geq 0, \qquad c(0) > 0. \tag{1b}$$

Then, the solution of (1) exhibits an interior layer of "cusp"-type at the simple interior turning point $x = 0$. It is well known (see e.g. [3], [4, p. 71], [6, Lemma 2.3]) that the derivatives of the solution can be bounded by

$$\left|u^{(i)}(x)\right| \leq C\left(1 + \left(\varepsilon^{1/2} + |x|\right)^{\lambda - i}\right) \tag{2}$$

where the parameter λ satisfies $0 < \lambda < \bar{\lambda} := c(0)/|a'(0)| = c(0)/b(0)$. If $\bar{\lambda}$ is not an integer, the estimate even holds for $0 < \lambda \leq \bar{\lambda}$, see references cited above.

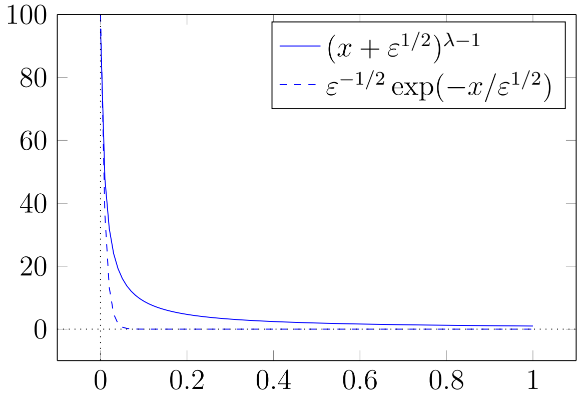

Fig 1. First derivatives of different layer types ($\varepsilon = 10^{-4}$, $\lambda = 10^{-2}$)

A common strategy to enable ε-uniform estimates for singularly perturbed problems is the use of layer-adapted meshes to handle the occuring layers. We will shortly present two suitable meshes for layers of "cusp"-type in the following. Besides their definition we give error estimates in the energy norm for higher order finite elements on these meshes. Moreover, some numerical results are presented.

Throughout the paper let C denote a positive generic constant that is independent of ε and the number of mesh points. For spaces, norms, and inner products standard notation is used, e.g. $\|\cdot\|$ is the L^2 norm and $(\cdot,\cdot)$ the L^2 inner product.

2. Meshes for layers of "cusp"-type

Many researchers have studied layer-adapted meshes in the last decades. Particularly meshes for exponential layers have been examined a lot. Since these layers fade away very quickly it is possible to use meshes that are fine in the layer regions only, such as S-type meshes. Unfortunately, layers of "cusp"-type behave much different (for an illustration see Fig. 1). They are "much wider" than $\mathcal{O}(\varepsilon)$. Indeed, if $\lambda < 1$ we can not guarantee $|u'| \le C$ outside of $[-\varepsilon^\theta, \varepsilon^\theta]$ for any fixed positive constant θ. Therefore, in order to capture the layer, local refinements do not suffice.

In the following let $\lambda \in (0, k+1)$ with $k \ge 1$ which is the most difficult case. Thinking on k as the ansatz order of a finite element space otherwise all crucial derivatives of the solution could be bounded by a generic constant independent of ε. This would enable to prove optimal order ε-uniform estimates with standard methods on uniform meshes.

2.1. Graded meshes of Liseikin

At first we want to present special graded meshes which were used by Liseikin [4] to prove the uniform first order convergence of an upwind scheme for problem (1). His basic idea is to find a transformation $\varphi(\xi, \varepsilon)$ that eliminates the singularities of the solution when it is studied with respect to ξ. Condensing this approach for our layer type yields the

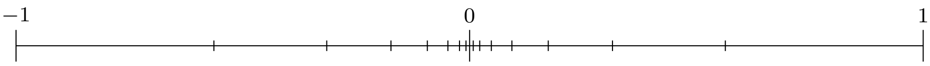

Fig 2. Graded mesh for $\varepsilon = 0.0001$, $\alpha = 0.01$, $N = 8$

task to find $\varphi : [0,1] \to [0,1]$ such that

$$\varphi' \left(\varphi + \varepsilon^{1/2}\right)^{\lambda-1} \le C, \qquad \varphi(0) = 0, \qquad \varphi(1) = 1. \tag{3}$$

As result we gain the mesh generating function

$$\varphi(\xi,\varepsilon) = \begin{cases} \left(\varepsilon^{\alpha/2} + \xi\left[(1+\varepsilon^{1/2})^\alpha - \varepsilon^{\alpha/2}\right]\right)^{1/\alpha} - \varepsilon^{1/2} & \text{for } 0 \le \xi \le 1, \\ \varepsilon^{1/2} - \left(\varepsilon^{\alpha/2} - \xi\left[(1+\varepsilon^{1/2})^\alpha - \varepsilon^{\alpha/2}\right]\right)^{1/\alpha} & \text{for } 0 \ge \xi \ge -1, \end{cases} \tag{4}$$

where $0 < \alpha \le \lambda$. Here α serves as grading parameter. Note that by construction $\varphi(0,\varepsilon) = 0$ and $\varphi(\pm 1,\varepsilon) = \pm 1$. The mesh points are then generated by $x_i = \varphi(\frac{i}{N},\varepsilon)$, $i = -N,\dots,N$. We define the mesh interval lengths by $h_i := x_i - x_{i-1}$ and set $h := N^{-1}$.

An easy calculation shows that condition (3) is satisfied by (4). We have, cf. [2],

$$\frac{\partial\varphi}{\partial\xi}\left(\varphi + \varepsilon^{1/2}\right)^{\lambda-1} \le C\frac{(1+\varepsilon^{1/2})^\alpha - \varepsilon^{\alpha/2}}{\alpha} \le C\min\left\{\alpha^{-1}, 1 + \left|\log_2(\varepsilon^{1/2})\right|\right\}$$

where we, in general, can not prevent the dependence on α. Since φ is associated to the mesh points and $h_i = h\frac{\partial\varphi}{\partial\xi}(\xi_i,\varepsilon)$ for a $\xi_i \in (x_{i-1}, x_i)$ by the mean value theorem, the above property of the mesh generating function yields the following lemma. We only consider $\xi \ge 0$ due to symmetry.

Lemma 1 (see [2, Lemma 2.1])**.** *Let* $\lambda > 0$ *and* $0 < \alpha \le \min\{\lambda/k, 1\}$ *with* $k \in \mathbb{N}$, $k \ge 1$ *then*

$$h_i^k\left(x_{i-1} + \varepsilon^{1/2}\right)^{\lambda-k} \le \begin{cases} Ch^k & \textit{for } 2 \le i \le N, \\ h_1^k \varepsilon^{(\lambda-k)/2} \le Ch^k & \textit{for } i = 1, \quad \varepsilon \ge h^{2/\alpha}. \end{cases}$$

If $0 < \alpha \le 1/(2k)$ *with* $k \in \mathbb{N}$, $k \ge 1$ *and* $\varepsilon \le h^{2/\alpha}$ *then we have*

$$x_1 \le Ch^{2k}.$$

In general, we have for $0 < \alpha \le 1$

$$h_i \le Ch \qquad \textit{for } 1 \le i \le N.$$

The constants C in Lemma 1 may depend on α and k. Note that there are two characteristic cases. In the first one, we can bound a term of the form $h_i^k(x_{i-1} + \varepsilon^{1/2})^{\lambda-k}$ by CN^{-k}. In the other case, we know that the length of the mesh interval next to the turning point can be bounded by CN^{-2k}.

2.2. Piecewise equidistant meshes of Sun and Stynes

As a second approach we want to present the meshes of Sun and Stynes [6, Section 5.1]. They generalise the basic idea of Shishkin and propose a mesh which is equidistant in each of $\mathcal{O}(\ln N)$ subintervals.

For $\varepsilon \in (0,1]$ and given positive integer N we set

$$\sigma = \max\left\{\varepsilon^{(1-\lambda/(k+1))/2}, N^{-(2k+1)}\right\} \qquad \text{and} \qquad \mathcal{K} = \left\lfloor 1 - \frac{\ln(\sigma)}{\ln(10)}\right\rfloor,$$

where $\lfloor z \rfloor$ denotes the largest integer less or equal to z. The mesh is constructed in two steps: First the interval $(0,1]$ is partitioned in a logarithmic sense into the $\mathcal{K}+1$ subintervals $(0,10^{-\mathcal{K}}]$, $(10^{-\mathcal{K}}, 10^{-\mathcal{K}+1}], \ldots, (10^{-1}, 1]$. Then each of these subintervals is divided uniformly into $\lfloor N/(\mathcal{K}+1)\rfloor$ parts.

It is easy to see that

$$\mathcal{K}+1 \le 2 + \min\left\{\frac{1-\lambda/(k+1)}{2}\frac{|\ln(\varepsilon)|}{\ln(10)}, (2k+1)\frac{\ln(N)}{\ln(10)}\right\} \le C\ln N. \tag{5}$$

Consequently, for N sufficiently large, we have $\mathcal{K}+1 \le N$. For simplicity we assume that $\lfloor N/(\mathcal{K}+1)\rfloor = N/(\mathcal{K}+1)$.

Lemma 2 (see [1, Lemma 3.1])**.** *Let $j = 0,1$. The following inequalities hold*

$$h_i^{k+1-j}\left(x_{i-1}+\varepsilon^{1/2}\right)^{\lambda-(k+1-j)} \le C\left((\mathcal{K}+1)N^{-1}\right)^{k+1-j}, \quad \text{for} \quad x_i \in (10^{-\mathcal{K}}, 1], \tag{6}$$

$$h_i^{k+1-j}\left(x_{i-1}+\varepsilon^{1/2}\right)^{\lambda-(k+1-j)} \le C\,(i-1)^{-(k+1-j)}, \quad \text{for} \quad x_i \in (x_1, 10^{-\mathcal{K}}]. \tag{7}$$

If $\sigma = \varepsilon^{(1-\lambda/(k+1))/2}$, then

$$h_i^{k+1-j}\left(x_{i-1}+\varepsilon^{1/2}\right)^{\lambda-(k+1-j)} \le C\left((\mathcal{K}+1)N^{-1}\right)^{k+1-j}, \quad \text{for} \quad x_i \in (0, 10^{-\mathcal{K}}]. \tag{8}$$

In general, the mesh interval length can be bounded by

$$h_i \le (\mathcal{K}+1)N^{-1}.$$

Furthermore, in the case of $\sigma = N^{-(2k+1)}$, we have

$$x_1 = h_1 \le (\mathcal{K}+1)N^{-2(k+1)}. \tag{9}$$

Similar to Lemma 1 we have two characteristic cases. In the fist one we can estimate terms of the form $h_i^k\left(x_{i-1}+\varepsilon^{1/2}\right)^{\lambda-k}$ by inequalities (6) and (8). In the second case we have a bound for x_1. Inequalities (7) and (9) provide information for the whole subinterval next to the turning point due to the piecewise uniformity.

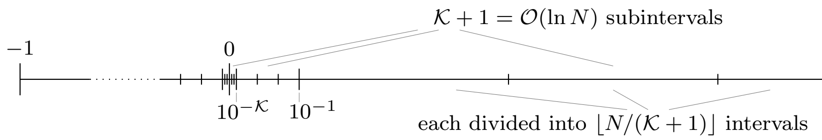

Fig 3. Typical piecewise equidistant mesh of Sun and Stynes

3. Error estimates for higher order finite elements

In this section we present the energy norm error estimates for finite elements of order $k \geq 1$. Without loss of generality (cf. [6, Lemma 2.1]) we may assume that

$$\left(c - \tfrac{1}{2}a'\right)(x) \geq \gamma > 0 \qquad \text{for all } x \in [-1,1], \quad \varepsilon \text{ sufficiently small.} \tag{10}$$

The standard finite element formulation of problem (1) reads as follows:

Find $u_N \in V^N$ such that

$$B_\varepsilon(u_N, v_N) = (f, v_N) \quad \text{for all } v_N \in V^N \tag{11}$$

where the bilinear form $B_\varepsilon(\cdot,\cdot) : H_0^1(-1,1) \times H_0^1(-1,1) \to \mathbb{R}$ is defined by

$$B_\varepsilon(v,w) := (\varepsilon v', w') + (av', w) + (cv, w)$$

and the trial and test space $V^N \subset H_0^1(-1,1)$ is given by

$$V^N := \left\{v \in C([-1,1]) : v|_{(x_{i-1},x_i)} \in P_k(x_{i-1},x_i)\,\forall i,\ v(-1) = v(1) = 0\right\}.$$

Here $P_k(x_a, x_b)$ denotes the space of polynomial functions of maximal order k over (x_a, x_b). Thanks to (10) the bilinear form $B_\varepsilon(\cdot,\cdot)$ is uniformly coercive over $H_0^1(-1,1) \times H_0^1(-1,1)$ in terms of the weighted energy norm $|||\cdot|||_\varepsilon$ defined by

$$|||v|||_\varepsilon := \left(\varepsilon\|v'\|^2 + \|v\|^2\right)^{1/2}.$$

In order to estimate the error of the finite element solution we split it as

$$u - u_N = (u - u_I) + (u_I - u_N)$$

where $u_I \in V^N$ denotes the standard Lagrange-interpolant of u. Using the coercivity and orthogonality of $B_\varepsilon(\cdot,\cdot)$ together with Cauchy Schwarz' inequality and the special structure of a, we obtain the following.

Lemma 3 (see [1, Lemma 2.1])**.** *Let u be the solution of* (1) *and u_N the solution of* (11) *on an arbitrary mesh. Then we have*

$$|||u_I - u_N|||_\varepsilon \leq C\Big(|||u_I - u|||_\varepsilon + \|x(u_I - u)'\|\Big).$$

Because of this estimate it remains to bound the interpolation error measured in different norms and semi norms only. For the meshes presented in Section 2. this can be done as in [2, Lemma 3.3 and 3.4] and [1, Lemma 3.2]. Finally, we obtain the error estimates in the energy norm.

Theorem 4 (see [2, Theorem 3.5]). *Let u be the solution of* (1) *and u_N the solution of* (11) *on the graded mesh of Section 2.1. with* $0 < \alpha \le \min\{\lambda/(k+1), 1/(2(k+1))\}$. *Then we have*

$$|||u - u_N|||_\varepsilon \le CN^{-k}.$$

Theorem 5 (see [1, Theorem 3.3]). *Let u be the solution of* (1) *and u_N the solution of* (11) *on the piecewise equidistant mesh of Section 2.2.. Then we have*

$$|||u - u_N|||_\varepsilon \le C\left((\mathcal{K}+1)N^{-1}\right)^k \le C\left(N^{-1}\ln N\right)^k.$$

Note that the graded mesh of Liseikin seems to be optimal in a certain way for layers of "cusp"-type, i.e., there is no additional logarithmic factor in the error estimate. However, the constant in Theorem 4 may depends on α.

4. Numerical experiments

In this section we want to compare the two presented mesh types numerically. In order to do this we use the following test problem from [6].

Example 6. *Consider the problem*

$$\begin{aligned} -\varepsilon u'' - x(1+x^2)u' + \lambda(1+x^3)u &= f, \qquad \textit{for} \quad x \in (-1,1), \\ u(-1) = u(1) &= 0, \end{aligned}$$

where the right-hand side $f(x)$ is chosen such that the solution $u(x)$ is given by

$$u(x) = \left(x^2+\varepsilon\right)^{\lambda/2} + x\left(x^2+\varepsilon\right)^{(\lambda-1)/2} - (1+\varepsilon)^{\lambda/2}\left(1 + x\,(1+\varepsilon)^{-1/2}\right).$$

The parameter λ in Example 6 coincides with the quantity $\bar{\lambda} = c(0)/|a'(0)|$. Besides the derivatives of the solution behave like (2) and, thus, as good and as bad as assumed in theory.

For all computations we have used a FEM-code based on $\mathbb{SOFE}$ by Lars Ludwig [5]. The grading parameter α for the graded mesh is chosen as

$$\alpha = \min\{\lambda/(k+1), 1/(2(k+1))\}.$$

This is the largest possible choice allowed by Theorem 4. Note that numerical experiments suggest that a smaller choice of α has nearly no influence on the error, see [2, Figure 2].

In Figure 4 the energy norm error for finite elements of order $k = 1, \dots, 4$ calculated on the graded mesh (square marks) and on the piecewise equidistant mesh (triangle marks) applied to Example 6 with $\varepsilon = 10^{-8}$ and $\lambda = 0.005$ is plotted. By comparison to the given reference curves the proven error behaviour can be confirmed.

The logarithmic factor occurring in Theorem 5 can not be seen numerically. This is not surprising due to the dominance of the first term in the minimum of (5) for the studied ε and λ, see also [1, Remark 3.4]. Nevertheless, the error on the piecewise equidistant mesh is larger than the error on the graded mesh for all ansatz orders. So the graded meshes also seem to be "better" with respect to the magnitude of the energy norm error. For more numerical results we refer to [1, 2].

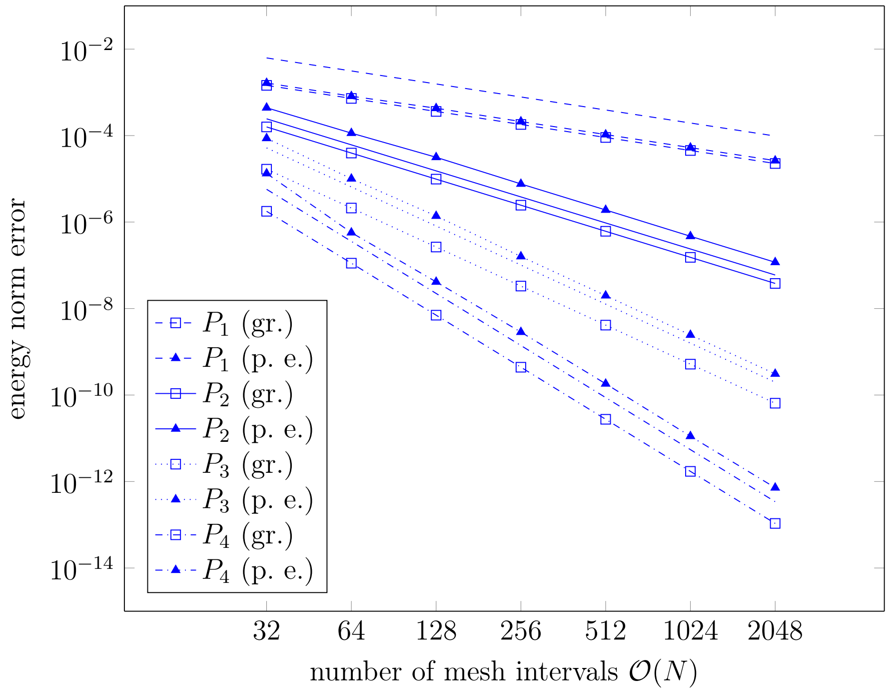

Fig 4. Energy norm error for P_k-FEM, $k = 1, \ldots, 4$ applied to Example 6 with $\varepsilon = 10^{-8}$ and $\lambda = 0.005$. Reference curves of the form $\mathcal{O}(N^{-k})$

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Бехер С., "МКЭ-анализ на адаптированных к слою сетках в задачах с точкой поворота, имеющих внутренний слой", *Моделирование и анализ информационых систем*, **23**:3 (2016), 240–247.

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Аннотация. Рассматриваются сингулярно возмущенные задачи с точкой поворота, решения которых имеют внутренний слой. Представлены подходящие для таких задач два типа адаптированных к слою сеток. Для обоих типов даны равномерные оценки погрешности в ε-весовой энергетической норме для конечных элементов высокого порядка. В целях сравнения этих сеток и подтверждения теоретических выводов использованы численные эксперименты.

Статья публикуется в авторской редакции.

Ключевые слова: сингулярные возмущения, точка поворота, внутренний слой, адаптированные к слою сетки, конечные элементы высокого порядка

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