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Convergence of the Difference Solutions of a Dirichlet Problem With a Discontinuous Derivative of the Boundary Function for a Singularly Perturbed Convection-Diffusion Equation

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Abstract. We consider a Dirichlet problem for a singularly perturbed convection-diffusion equation with constant coefficients in a rectangular domain in the case when the convection is parallel to the horizontal faces of the rectangular and directed to the right while the first derivative of the boundary function is discontinuous on the left face. Under these conditions the solution of the problem has a regular boundary layer in the neighborhood of the right face, two characteristic boundary layers near the top and bottom faces, and a horizontal interior layer due to the non-smoothness of the boundary function. We show that on the piecewise uniform Shishkin meshes refined near the regular and characteristic layers, the solution given by the classical five-point upwind difference scheme converges uniformly to the solution of the original problem with almost first-order rate in the discrete maximum norm. This is the same rate as in the case of a smooth boundary function. The numerical results presented support the theoretical estimate. They show also that in the case of the problem with a dominating interior layer the piecewise uniform Shishkin mesh refined near the layer decreases the error and gives the first-order convergence.

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Keywords: convection-diffusion, singular perturbation, interior layer, uniform convergence

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Setting the problem

We consider a Dirichlet problem for singularly perturbed convection-diffusion equation in a rectangle domain $\Omega = (0,1)\times(-1,1)$ with the boundary $\partial\Omega$:

$$\begin{aligned} Lu \equiv -\,\varepsilon\triangle u + a\,\partial u/\partial x + qu = f(x,y), \quad (x,y)\in\Omega, \quad \varepsilon\in(0,1], \\ a = const > 0, \qquad q = const > 0, \\ u = g(x,y), \quad (x,y)\in\partial\Omega. \end{aligned} \tag{1}$$

Assume that $g(x,y)$ on $\partial\Omega\setminus\{(0,0)\}$ and $f(x,y)$ on Ω are sufficiently smooth and

$$\frac{\partial g_1(y)}{\partial y}\Big|_{y=+0} \neq \frac{\partial g_1(y)}{\partial y}\Big|_{y=-0} \tag{2}$$

where $g_1(y) = g(0,y)$. It is known that a solution of the problem for a small ε can have a regular layer $O(\varepsilon)$ wide near the boundary $x = 1$, through which the flow is leaving the domain; the characteristic layers of the wideness $O(\sqrt{\varepsilon})$ near the boundaries $y = \pm 1$ parallel to the flow; corner layers near the vertices at the exit of the flow, and also corner singularities because no compatibility conditions at the corners of the domain are assumed.

1. Difference problem

In the domain Ω we define the following mesh $\overline{\Omega}^h$ as a direct product of one-dimensional meshes $\overline{\omega}_1^h(x)$ and $\overline{\omega}_2^h(y)$, where $\overline{\omega}_1^h(x) = \{x_i \,|\, 0 = x_0 < x_1 < \cdots < x_N = 1\}$, $\overline{\omega}_2^h(y) = \{y_i \,|\, -1 = y_{-N} < y_{-N+1} < \cdots < y_{-1} < y_0 = 0 < y_1 < \cdots < y_N = 1\}$.

Defining the mesh domain, we shall use a piece-wise uniform Shishkin mesh refining near the boundary $x = 1$ where the solution has a regular layer, and near the boundaries $y = \pm 1$, where it can have characteristic layers.

We introduce the following notation: steps of the mesh $h_{1,i} = x_i - x_{i-1}$, $h_{2,j} = y_j - y_{j-1}$ and $\hbar_{k,i} = (h_{k,i} + h_{k,i+1})/2$; divided diferences $v_{\bar{x},i,j} = (v_{i,j} - v_{i-1,j})/h_{1,i}$, $v_{x,i,j} = v_{\bar{x},i+1,j}$ and $v_{\hat{x},i,j} = (v_{i+1,j} - v_{i,j})/\hbar_{1,i}$; boundaries of the mesh domain $\partial\Omega^h = \overline{\Omega}^h \cap \partial\Omega$.

We approximate the problem for $u(x,y)$ by the classical five-point difference scheme for $u^h_{i,j}$ on the mesh $\overline{\Omega}^h$:

$$\begin{aligned} L^h u^h_{i,j} \equiv -\varepsilon(u^h_{\bar{x}\,\hat{x}} + u^h_{\bar{y}\,\hat{y}})_{i,j} + a\,u^h_{\bar{x}\,i,j} + qu^h_{i,j} = f(x_i,y_j), \qquad (x_i,\,y_j)\in\Omega^h = \overline{\Omega}^h\cap\Omega, \\ u^h_{i,j} = g(x_i,y_j), \qquad (x_i,y_j)\in\partial\Omega^h. \end{aligned} \tag{3}$$

Observe that the maximum principle holds for the difference problem in question.

For sufficiently smooth on the faces of the rectangle boundary functions the problem was considered by several authors, in particular in [1], [2], [3]. In the case when no compatibility conditions, except for minimal ones, are assumed in the corners of the domain in [2] on the piece-wise uniform Shishkin meshes refining near the regular and the characteristic layers it was obtained the convergence of the mesh solution to the solution of the original problem with rate $O(N^{-1}\ln^2 N)$ uniformly in ε (N is the number of the points in the mesh in every direction).

For the singularly perturbed convection–diffusion equation in a half-plane the problem with non-smooth boundary conditions, when there is a discontinuity of the boundary function or its derivatives, is considered in [4] where the estimates for the solution and itsderivatives depending on the parameter ε are given. We use these estimates, and also rely on the results of [2] and [3].

Below c denotes a positive constant independent of ε and N.

2. Decomposition of the solution

To begin with we single out a solution $u_1(x, y)$ related to the singularity of the boundary condition $g_1(y)$. For this purpose we define the function $g_1^*(y) = g_1(y) * \eta(y)$ on the line $\mathbb{R} = \{(x, y) \,|\, x = 0\}$ where

$$\eta(y) = \begin{cases} 1, & |y| \leqslant 1/3, \\ 0, & |y| \geqslant 2/3, \end{cases} \qquad \eta(y) \in C^\infty(\mathbb{R}).$$

We consider a bounded solution $u_1^*(x, y)$ of the problem

$$Lu_1^* \equiv -\varepsilon\triangle u_1^* + a\,\partial u_1^*/\partial x + qu_1^* = 0, \quad (x, y) \in \mathbb{R}_+^2, \qquad u_1^*(0, y) = g_1^*(y) \tag{4}$$

in the half-plane $\mathbb{R}_+^2 = \{(x, y), x > 0\}$. To estimate the solution $u_1^*(x, y)$ we use the results of [4]. The main theorem of this work in particular states the following. Let $r = \sqrt{x^2 + y^2}$, and let $\mathbb{R}_+$ (resp. $\mathbb{R}_-$) denote the interval $(0, \infty)$ (resp. $(-\infty, 0)$,).

Theorem 1. *Let $g_1^*(y) \in H^7(\mathbb{R}_+ \bigcup \mathbb{R}_-)$. Then there exists a constant c such that for $0 < \varepsilon < 1$ and $m = 0, 2, 3,\ \ n = 0, 1, 2, 3$ the following inequalities hold for the solution $u_1^*(x, y)$:*

$$\begin{aligned}
|D_x\, u_1^*(x, y)| &\leqslant c(1 + \ln r), \\
|D_x^m\, u_1^*(x, y)| &\leqslant c(1 + r^{-m+1}) \quad \text{for } r \leqslant 2\varepsilon, \\
|D_y^n\, u_1^*(x, y)| &\leqslant c(1 + r^{-n+1}) \quad \text{for } r \leqslant 2\varepsilon, \\
|D_x^m\, u_1^*(x, y)| &\leqslant c(1 + \sqrt{\varepsilon}\, r^{-m+1/2}\, e^{-cy^2/\varepsilon r} + r^{-m+1}\, e^{-cr/\varepsilon}) \quad \text{for } 2\varepsilon \leqslant r \leqslant \sqrt{2}, \\
|D_y^n\, u_1^*(x, y)| &\leqslant c(1 + \varepsilon^{(-n+1)/2}\, r^{(-n+1)/2}\, e^{-cy^2/\varepsilon r} + r^{-n+1}\, e^{-cr/\varepsilon}) \quad \text{for } 2\varepsilon \leqslant r \leqslant \sqrt{2}.
\end{aligned}$$

We represent the solution $u(x, y)$ of the original problem in the form $u = u_1 + u_2$ where u_1 is the restriction of the solution $u_1^*(x, y)$ to $\overline{\Omega}$ and $u_2(x, y)$ is a solution of the problem

$$\begin{aligned}
-\varepsilon\triangle u_2 + a\,\partial u_2/\partial x + qu_2 &= f(x, y), \qquad (x, y) \in \Omega, \\
u_2(x, y)|_{\partial\Omega} &= (g(x, y) - u_1(x, y))|_{\partial\Omega}.
\end{aligned}$$

The boundary function of the solution $u_2(x, y)$ has no singularity on $x = 0$.

3. The estimate of the convergence rate of the mesh solution

According to the decomposition $u = u_1 + u_2$ we write down the difference solution as $u^h = u_1^h + u_2^h$ where every u_k^h is a solution of the problem

$$L^h u_{k,i,j}^h = Lu_k(x_i, y_j), \qquad (x_i, y_j) \in \Omega^h, \qquad u_{k,i,j}^h|_{\partial\Omega^h} = u_k(x_i, y_j)|_{\partial\Omega^h}. \tag{5}$$

It is our main task to investigate the convergence rate of the difference solution u_1^h to $u_1(x,y)$.

The approximation error of the equation on the solution $u_1(x,y)$ is equal to $\Psi_{i,j}(u_1) = L^h(u_{1,i,j} - u^h_{1,i,j})$. The estimate $\Psi_{i,j}(u_1)$ is obtained from the estimates of the the solution $u_1(x,y)$ and some additional inequalities (similar to how it was done in [2]):

$$|L^h(u_1 - u_1^h)|_{i,j} = |\Psi_{i,j}(u_1)| \leqslant c\, \left(\varepsilon(h_{1,i+1}\, r_{i,j}^{-2} + h_{2,j+1}\, r_{i,j}^{-2}) + h_{1,i}\, r_{i,j}^{-1} + h_{2,j+1}\, r_{i,j}^{-1}\right). \quad (6)$$

For the estimation of the $|u_1 - u_1^h|$ we use the barrier function introduced by V.B.Andreev in [2]: $B(r',\varphi') = \ln\dfrac{r'}{H} + (\pi/2-\varphi')(\pi/2+\varphi'+1)+1$ where $r' = \sqrt{x'^2+y^2}$, $\varphi' = \arctan\dfrac{y}{x'}$, $x' = x + bH$, $b \geqslant 1$. The following lemma is an analog of the Theorem 1 of that work for the mesh we consider here.

Lemma 1. *If the function $w^h_{i,j}$ satisfies the following inequalities*

$$|L^h w^h_{i,j}| \leqslant c\,(\varepsilon r_{ij}^{-2} + r_{ij}^{-1} + 1) \quad \textit{for} \quad (x_i,y_j)\in\Omega^h, \qquad |w^h_{i,j}| \leqslant c \quad \textit{for} \quad (x_i,y_j)\in\partial\Omega^h,$$

then the estimate $|w^h_{i,j}| \leqslant c\ln N$ is fulfilled in the whole domain $\overline{\Omega}^h$.

The following estimate of the error of the solution u_1^h of the difference problem follows from (6) and Lemma 1 :

Lemma 2. *Let $u_1(x,y)$ be the above defined restriction of the solution of the problem (4) to the domain $\overline{\Omega}$, and u_1^h- be the solution of the problem (5). Let $f(x,y)$ be smooth enough in Ω, the same holds for $g_(x,y)$ in $\partial\Omega\setminus\{(0,0)\}$, and $g(0,y)\in H^7(\{-2/3\leqslant y < 0\}\bigcup\{0<y\leqslant 2/3\})$. Then the following estimate holds*

$$|u_1 - u_1^h| \leqslant c\,N^{-1}\ln^2 N, \quad (x_i,y_j)\in\overline{\Omega}^h.$$

A convergence rate for the solution u_2^h is obtained in [2], [3]:

$$|u_2 - u_2^h| \leqslant c\,N^{-1}\ln^2 N, \quad (x_i,y_j)\in\overline{\Omega}^h.$$

The last two estimates imply our main result:

Theorem 2. *Let $u(x,y)$ be a solution of the problem (1), (2). Let $f(x,y)$ be smooth enough in Ω, the same holds for $g_(x,y)$ in $\partial\Omega\setminus\{(0,0)\}$ and $g_1(y)\in H^7\big(\{-2/3\leqslant y< 0\}\bigcup\{0<y\leqslant 2/3\}\big)$. Then the following inequality holds for the solution $u^h_{i,j}$ of the problem (3) on the mesh $\overline{\Omega}^h$ uniformly in ε:*

$$|u(x_i,y_j) - u^h_{i,j}|_{L^h_\infty} \leqslant c\,N^{-1}\ln^2 N, \qquad (x_i,y_j)\in\overline{\Omega}^h.$$

4. Example of numerical solution

We consider the following problem in the unit square:

$$-\varepsilon\triangle u + 2\,\partial u/\partial x + 3u = f(x,y), \qquad (x,y)\in\Omega=(0,1)^2,$$
$$u(x,y)=0, \qquad (x,y)\in\partial\Omega\setminus\{(0,y)\},$$
$$u(0,y) = \begin{cases} y^3, & 0<y\leqslant 0.5,\\ (1-y)^3, & 0.5\leqslant y\leqslant 1.\end{cases}$$

The solution of this problem has a discontinuity of the first derivative on the boundary $x = 0$ at $y = 0.5$, thus it has a weak interior layer. For $f(x, y) = 0$ the solution has relatively small regular layer near the boundary $x = 1$, no characteristic layers along the boundaries $y = 0$, $y = 1$ and singularities in the corners of the square (Fig.1). The problem of the regular layer is resolved by means a mesh refining in the strip of width $\sigma_1 = \min\{1/2\,\varepsilon \ln N;\ 1/4\}$.

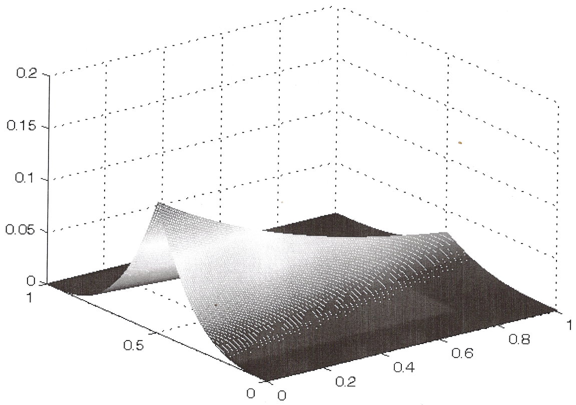

Fig. 1. Solution

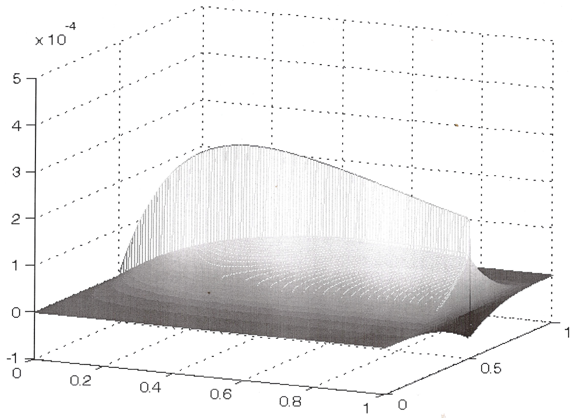

Fig. 2. The error of the solution

To estimate the convergence rate we use the values of $e_1 = \max\limits_{(i,j)} \left| u_N^h(i, j) - u_{2N}^h(2i, 2j) \right|$ where $u_N^h(i, j)$ is a difference solution on the mesh with the number of points equal to N in every direction and u_{2N}^h is a solution on the mesh with the double number of points while the value of σ_1 is the same. In the Table 1 the e_1, $e_2 = e_1 \cdot N$, $e_3 = e_1 \cdot N/\ln N$ for every considered value ε are evaluated.

The numerical investigations show that the maximal errors in the solution are located in the interior layer, near the middle of it (Fig. 2).

For $\varepsilon = 10^{-2}$ the convergence rate is $O(N^{-1})$; for $\varepsilon \leqslant 10^{-6}$ it is $O(N^{-1} \ln N)$ which meets the above obtained estimate (Lemma 2). In course of decreasing the ε the error stabilizes which gives evidence of the uniform in ε convergence. The piecewise uniform Shishkin mesh with $\sigma_2 = \min\{q^{-1/2} \sqrt{\varepsilon} \ln N;\ 1/8\}$ refined along the interior layer improves the convergence up to the rate $O(N^{-1})$ for our solution with the dominating interior layer.

If for the right hand part of the equation we have $f(x, y) = 1$ then the solution has a more serious regular interior layer and characteristic layers. In this case the maximal error is located in the neighborhood of the regular interior layer. Moreover, the convergence rate is $O(N^{-1} \ln^2 N)$ uniformly in ε which meets the above obtained estimates (Theorem 2).

The author is grateful to V.B. Andreev for setting the problem, his attention and valuable remarks.

Table 1

The errors of solution $e_1 = \max |u_N^h - u_{2N}^h|$
and it compositions on N and N/lnN

N	32	64	128	256
		$\varepsilon = 10^{-2}$		
e_1	$3.15{\cdot}10^{-3}$	$1.63{\cdot}10^{-3}$	$0.80{\cdot}10^{-3}$	$0.39{\cdot}10^{-3}$
e_2	$1.01{\cdot}10^{-1}$	$1.04{\cdot}10^{-1}$	$1.02{\cdot}10^{-1}$	$0.98{\cdot}10^{-2}$
e_3	$2.91{\cdot}10^{-2}$	$2.51{\cdot}10^{-2}$	$2.11{\cdot}10^{-2}$	$1.78{\cdot}10^{-2}$
		$\varepsilon = 10^{-4}$		
e_1	$1.49{\cdot}10^{-3}$	$1.10{\cdot}10^{-3}$	$0.70{\cdot}10^{-4}$	$0.37{\cdot}10^{-4}$
e_2	$4.76{\cdot}10^{-2}$	$7.05{\cdot}10^{-2}$	$8.95{\cdot}10^{-2}$	$9.41{\cdot}10^{-2}$
e_3	$1.37{\cdot}10^{-2}$	$1.69{\cdot}10^{-2}$	$1.84{\cdot}10^{-2}$	$1.70{\cdot}10^{-2}$
		$\varepsilon = 10^{-6}$		
e_1	$1.02{\cdot}10^{-3}$	$0.54{\cdot}10^{-4}$	$0.29{\cdot}10^{-4}$	$0.17{\cdot}10^{-4}$
e_2	$3.28{\cdot}10^{-2}$	$3.43{\cdot}10^{-2}$	$3.69{\cdot}10^{-2}$	$4.46{\cdot}10^{-2}$
e_3	$9.46{\cdot}10^{-3}$	$8.24{\cdot}10^{-3}$	$7.60{\cdot}10^{-3}$	$8.05{\cdot}10^{-3}$
		$\varepsilon = 10^{-7}$		
e_1	$1.02{\cdot}10^{-3}$	$0.53{\cdot}10^{-4}$	$0.27{\cdot}10^{-4}$	$0.14{\cdot}10^{-4}$
e_2	$3.26{\cdot}10^{-2}$	$3.36{\cdot}10^{-2}$	$3.43{\cdot}10^{-2}$	$3.54{\cdot}10^{-2}$
e_3	$9.41{\cdot}10^{-3}$	$8.07{\cdot}10^{-3}$	$7.07{\cdot}10^{-3}$	$6.39{\cdot}10^{-3}$
		$\varepsilon = 10^{-8}$		
e_1	$1.02{\cdot}10^{-3}$	$0.52{\cdot}10^{-4}$	$0.27{\cdot}10^{-4}$	$0.13{\cdot}10^{-4}$
e_2	$3.26{\cdot}10^{-2}$	$3.35{\cdot}10^{-2}$	$3.40{\cdot}10^{-2}$	$3.44{\cdot}10^{-2}$
e_3	$9.40{\cdot}10^{-3}$	$8.06{\cdot}10^{-3}$	$7.01{\cdot}10^{-3}$	$6.20{\cdot}10^{-3}$

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Ершова Т. Я., "Сходимость сеточного решения задачи Дирихле с разрывной производной граничной функции для сингулярно возмущенного уравнения конвекции-диффузии", *Моделирование и анализ информационых систем*, **23**:3 (2016), 291–297.

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Аннотация. Рассмотрена задача Дирихле для сингулярно возмущенного уравнения конвекции-диффузии с постоянными коэффициентами в прямоугольнике в случае, когда конвекция параллельна горизонтальным сторонам прямоугольника и направлена в сторону правой границы, а на левой границе первая производная граничной функции разрывна. При этих условиях решение задачи имеет регулярный пограничный слой в окрестности правой границы, два характеристических пограничных слоя около верхней и нижней границы и горизонтальный внутренний слой, возникающий из-за малой гладкости граничной функции. Показано, что на кусочно равномерных сетках Шишкина, сгущающихся около регулярного и характеристических слоев, решение, получаемое по классической пятиточечной разностной схеме с направленной разностью, равномерно по малому параметру сходится к решению исходной задачи в сеточной норме максимум модуля почти с первым порядком, а именно с той же скоростью, что и при гладкой граничной функции. Представлены численные результаты, подтверждающие теоретическую оценку. Также показано, что в случае задачи с преобладающим внутренним слоем кусочно равномерная сетка Шишкина, сгущающаяся около внутреннего слоя, дает уменьшение ошибки и сходимость с первым порядком.

Статья публикуется в авторской редакции.

Ключевые слова: конвекция-диффузия, сингулярное возмущение, внутренний слой, сеточное решение, равномерная сходимость

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