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Analytic-Numerical Approach to Solving Singularly Perturbed Parabolic Equations with the Use of Dynamic Adapted Meshes

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Abstract. The main objective of the paper is to present a new analytic-numerical approach to singularly perturbed reaction-diffusion-advection models with solutions containing moving interior layers (fronts). We describe some methods to generate the dynamic adapted meshes for an efficient numerical solution of such problems. It is based on *a priori* information about the moving front properties provided by the asymptotic analysis. In particular, for the mesh construction we take into account *a priori* asymptotic evaluation of the location and speed of the moving front, its width and structure. Our algorithms significantly reduce the CPU time and enhance the stability of the numerical process compared with classical approaches.

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Keywords: singularly perturbed parabolic periodic problems, interior layer, Shishkin mesh, dynamic adapted mesh

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Introduction

Singularly perturbed parabolic problems often feature narrow boundary and interior layers (stationary or moving fronts). Their numerical treatment by means of difference schemes requires meshes with a very large number of nodes. In some cases it leads to unacceptable CPU-times and unreliable solutions due the accumulation of round-off errors. To overcome both problems we propose an effective asymptotic-numerical approach for problems with moving interior layers in nonlinear reaction-diffusion-advection equations. Its motivation comes from the following observations: the smaller the parameter ε in singularly perturbed problem, the more rough and unstable the constructed numerical solution we obtain; but the more precise *a priori* information about the exact solution we can get from the asymptotic analysis. So, an appropriate combination of asymptotic analysis and numerical schemes should improve the effectiveness of numerical calculations, increase its speed and stability.

This idea has been used recently for problems with stationary interior layers in [1, 2, 3, 4, 5, 6, 7], where special grids have been used. In the case of moving interior layers, fairly complicated difference schemes has been constructed in the papers [2, 8, 9, 10] and [11] where one example of periodic problem was considered.

In this paper we present an effective analytic-numerical approach for the numerical approximation of periodic solutions with moving interior layers in reaction-diffusion-advection equations. This approach exploits the asymptotic results obtained in [12, 13, 14] and is based on the construction of *dynamic adapted mesh*(DAM).

Note, that numerical investigation of time-periodic problems generates a number of specific features. The main of them is that there is no information about the location of the interior layer at the initial time moment. To determine the initial conditions, which are necessary for further numerical calculations with the dynamic adapted mesh, we can use asymptotic analysis of the problem.

Another approach for numerical solving of periodic problems is to use the method of relaxation count. However, in this case it is necessary to know the stability of the periodic solution and investigate its domain of influence for the correct choice of the initial approximation. This proof and related estimates also can be done using the asymptotic analysis of the periodic problem by the methods developed in [12, 13].

The paper is structured as follows. In Section 1. we discuss methods by which we can obtain *a priori* information that will be used for the process of constructing a DAM. In Section 2. we briefly describe the main ideas for constructing DAM.

1. Asymptotic analysis and *a priori* information

To demonstrate our approach we consider the following problem

$$\begin{cases} \varepsilon\left(\dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial u}{\partial t}\right) = A(u,x,t)\dfrac{\partial u}{\partial x} + B(u,x,t) \\ \text{for } (x,t) \in D := \{x \in (-1,1);\ t \in \mathbb{R}\}, \\ u(-1,t) = u_{left}(t), \quad u(1,t) = u_{right}(t) \quad \text{for } t \in \mathbb{R}, \\ u(x,0) = u(x,t+T) \qquad \text{for } x \in [-1,1],\ t \in \mathbb{R}, \end{cases} \tag{1}$$

where the parameter ε is sufficiently small ($0 < \varepsilon \ll 1$) and the functions $A(u,x,t)$, $B(u,x,t)$, $u_{left}(t)$ and $u_{right}(t)$ are sufficiently smooth and T-periodic in t.

The methods of asymptotic analysis for singularly perturbed time-periodic parabolic problems was developed in [12, 13]. It is known [13] that under certain conditions this problem has a solution of moving front type: in the interval $(-1,1)$ there is some periodically moving point $x_{tr}(t,\varepsilon)$ which is connected with a thin transition layer containing $x_{tr}(t,\varepsilon)$ such that the solution for $x < x_{tr}(t,\varepsilon)$ is close to some level $u_{left}(t)$ and for $x > x_{tr,\varepsilon}(t)$ close to some level $u_{right}(t)$, where $u_{left}(t) \neq u_{right}(t)$ for all t. The main purpose of this paper is to present an effective numerical method for the solution of the moving front type which is based on the asymptotic *a priori* information such as *location* and/or *speed* of the internal layer (front), *width* of the internal layer and *structure* of the internal layer. This information can be obtained by the asymptotic analysis of the problem (1) which was developed in [13]. Here we recall some ideas and formulas from [13].

If we put $\varepsilon = 0$ in (1) we get the reduced equation and define two functions

$$\begin{aligned} \varphi^l(x,t): \quad & A(u,x,t)\frac{du}{dx} + B(u,x,t) = 0, \quad u(-1,t) = u_{left}(t); \\ \varphi^r(x,t): \quad & A(u,x,t)\frac{du}{dx} + B(u,x,t) = 0, \quad u(1,t) = u_{right}(t), \end{aligned} \tag{2}$$

where t has to be considered as a parameter.

Condition 1. *Suppose that for $(x,t) \in \bar{D} := \{x \in [-1,1], t \in \mathbb{R}\}$ there exist T-periodic in t solutions $\varphi^l(x,t)$ and $\varphi^r(x,t)$ of the problem (2) satisfying the following inequalities for all $(x,t) \in \bar{D}$*

$$\begin{aligned} &a) \quad \varphi^l(x,t) < \varphi^r(x,t), \\ &b) \quad A(\varphi^l(x,t),x) > 0, \quad A(\varphi^r(x,t),x) < 0. \end{aligned} \tag{3}$$

Let us define the function

$$I(x,t) := \int_{\varphi^l(x,t)}^{\varphi^r(x,t)} A(u,x,t)du. \tag{4}$$

Condition 2. *The equation*

$$I(x,t) = 0 \tag{5}$$

has a T-periodic solution $x_0(t)$ satisfying for all $t \in \mathbb{R}$

$$\begin{aligned} &a) \; -1 < x_0(t) < 1, \\ &b) \int_{\varphi^l(x_0(t),t)}^{s} A(u,x_0(t),t)du > 0 \quad \text{for} \quad s \in \left(\varphi^l(x_0(t),t), \varphi^r(x_0(t),t)\right). \end{aligned} \tag{6}$$

Condition 3. *The solution $x_0(t)$ of equation (5) obeys the condition*

$$\frac{\partial I}{\partial x}(x_0(t),t) < 0 \quad \text{for all} \quad t \in \mathbb{R}. \tag{7}$$

In [13] under Conditions 1–3 the existence of the solution of (1) with moving internal layer was proved and rigorous asymptotic analysis of this solution was presented.

Asymptotic of the solutions of (1) was built in the form

$$U^{l,r}(x,t,\varepsilon) = \bar{u}^{l,r}(x,t,\varepsilon) + Q^{l,r}(\xi,t,\varepsilon) = \sum_{i=0}^{\infty} \varepsilon^i \left(\bar{u}_i^{l,r}(x,t) + Q_i^{l,r}(\xi,t) \right), \tag{8}$$

where $\bar{u}^{l,r}(x,t,\varepsilon)$ are regular functions which represent the solution far from the transition point $x_{tr}(t,\varepsilon)$; the functions $Q^{l,r}(\xi,t,\varepsilon)$, where $\xi = (x - x_{tr}(t,\varepsilon))/\varepsilon$, describe the moving front located near this point; $\xi \le 0$ is related to a function with the upper index l and $\xi \ge 0$ to a function with the upper index r.

Location of the transition point $x_{tr}(t,\varepsilon)$ presented in the form of a power series in ε

$$x_{tr}(t,\varepsilon) = x_0(t) + \varepsilon x_1(t) + ... \tag{9}$$

where $x_i(t)$, $i = 1, 2, \ldots$ are T-periodic functions. The terms of series in (8) and (9) can be determined by the asymptotic procedure [13] from $C^{(1)}$–matching conditions for functions $U^l(x,t,\varepsilon)$ and $U^r(x,t,\varepsilon)$ – continuous matching for functions and its first derivatives at the point $x = x_{tr}(t,\varepsilon)$ for all orders of ε.

It was also proved in [13] that functions $Q^{l,r}(\xi,t,\varepsilon)$ exponentially tend to zero if $\xi \to \pm\infty$. So the front width can be estimated as $d = C\varepsilon|\ln\varepsilon|$.

In Section 2. we use this information and illustrate our analytic-numerical approach by means of the following particular case of problem (1): $A(u,x,t) = -u$ and $B(u,x,t) = ub(t)$, where $b(t) = 2 + \cos(4\pi t)$; $u_{left}(t) = -8 + \sin(4\pi t)$, $u_{right}(t) = 8 - 2\sin(4\pi t)$.

For this example we have

$$\varphi^l(x,t) = -8 + \sin(4\pi t) + (x+1)\,(2 + \cos(4\pi t))\,;$$
$$\varphi^r(x,t) = 8 - 2\sin(4\pi t) + (x-1)\,(2 + \cos(4\pi t))\,.$$

Condition 1 is satisfied because it holds for all $x \in [-1, 1]$

$$\varphi^l(x,t) - \varphi^r(x,t) = -16 + 2\,(2 + \cos(4\pi t)) + 3\sin(4\pi t) < 0;$$

$$A(\varphi^l(x,t),x,t)) = 8 - (x+1)\,(2 + \cos(4\pi t)) - \sin(4\pi t) > 0,$$
$$A(\varphi^r(x,t),x,t) = -8 - (x-1)\,(2 + \cos(4\pi t)) + 2\sin(4\pi t) < 0.$$

The function $I(x,t)$ defined in (4) reads

$$I(x,t) = \int\limits_{\varphi^l(x,t)}^{\varphi^r(x,t)} -u du = \frac{1}{2}\,(2x\,(2 + \cos(4\pi t)) - \sin(4\pi t))\,(-16 + 2\,(2 + \cos(4\pi t)) + 3\sin(4\pi t))$$

and (5) gives the following expression for the zero order term of moving front position

$$x_0(t) = \frac{\sin(4\pi t)}{4 + 2\cos(4\pi t)} \in [-1; 1] \qquad \text{for all} \qquad t \in \mathbb{R}. \tag{10}$$

Conditions 2 and 3 hold true for this $x_0(t)$.

At first order of ε we have from [13]

$$x_1(t) = -\frac{1}{2b(t)} \cdot \left[2x_0'(t) + \left(\bar{u}_1^l(x_0(t), t) + \bar{u}_1^r(x_0(t), t)\right)\right], \tag{11}$$

where

$$\bar{u}_1^l(x,t) = \frac{d}{dt}\left[\frac{u_{left}(t)}{b(t)}\right] \cdot \ln\left(1 + \frac{b(t)}{u_{left}(t)}(x+1)\right) + \frac{d}{dt}\left[\ln(b(t))\right] \cdot (x+1);$$

$$\bar{u}_1^r(x,t) = \frac{d}{dt}\left[\frac{u_{right}(t)}{b(t)}\right] \cdot \ln\left(1 + \frac{b(t)}{u_{right}(t)}(x-1)\right) + \frac{d}{dt}\left[\ln(b(t))\right] \cdot (x-1).$$

2. Dynamic adapted mesh construction

Our idea of dynamic adaptive mesh construction is quite simple. If we know the width of the transition layer, we can introduce a basic uniform mesh with steps equal to this width. Then refine two intervals that are nearest to the transition point $x_{tr}(t, \varepsilon)$ which location we estimate by asymptotic analysis as $x_0(t) + \varepsilon x_1(t)$. Next, if we know the position of the transition layer for each time step, we can track whether the transition layer is located in these intervals or not. If the transition layer starts to leave one of these intervals, we refine the following or previous basic interval, perform an interpolation of the function on these additional nodes. In the following calculations we discard the nodes of the refined interval that are farthest from the position of the transition point. For an appropriate interpolation we need an information about the structure of the transition layer. As a result, we have again only two refined basic intervals. Some example of the constructed DAM by this approach is represented on Figure 1.

Another approach is to construct classical "Shishkin mesh"(see Figure 2).

A crucial assumption for this constructing process is the possibility to obtain a corresponding *a priori* information. This problem was discussed in Section 1. for one type of reaction-diffusion-advection equations.

Some example of numerical calculations is represented on the Figure 3.

3. Conclusion

Asymptotic analysis of a singularly perturbed problem gives the *a priori* which can be used for efficient mesh construction. This fact provides the possibility for a productive combination of asymptotic and numerical approaches in order to substantially improve the effectiveness of numerical calculations.

Based on these ideas we propose an efficient analytic-numerical algorithm for a singularly perturbed reaction-diffusion-advection equations that allows significantly to reduce the complexity and to enhance the stability of the numerical calculations in comparison with classical approaches. As a result, we can essentially save CPU time and significantly speed up the process of constructing approximate solutions with a suitable accuracy.

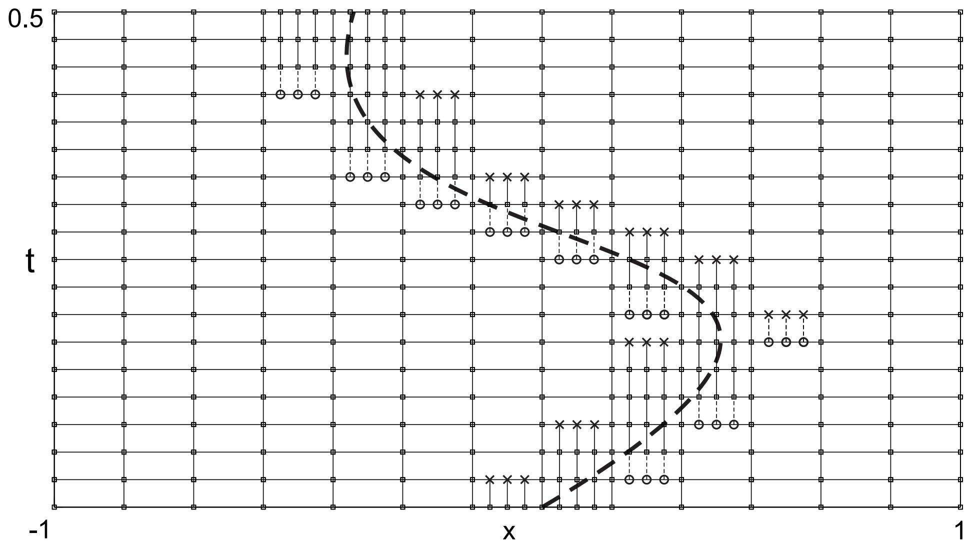

Fig 1. Some result of the process of dynamic adapted mesh construction: □ – node that is used for calculations; o – node in which function was interpolated; × – node that was discarded from the process of calculations on the following steps

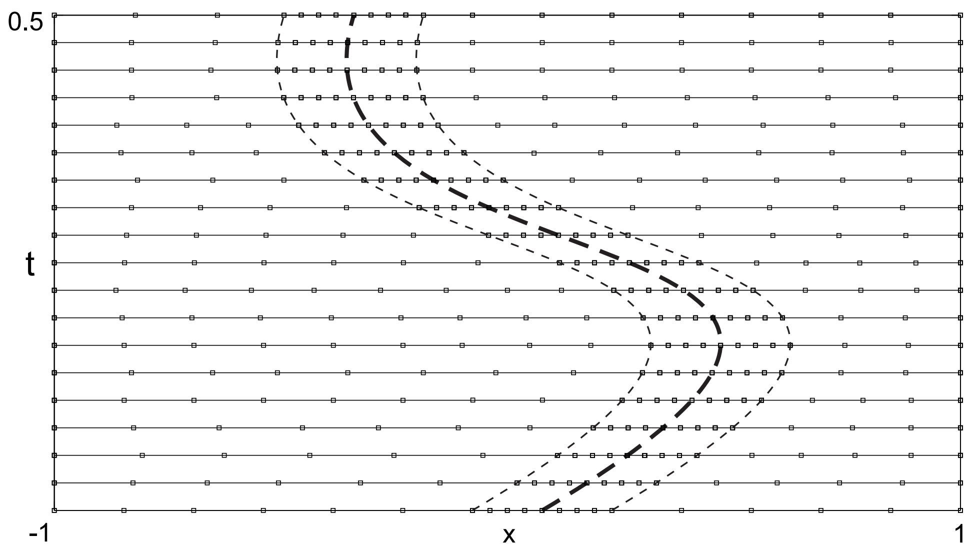

Fig 2. Some result of the process of dynamic adapted mesh construction in the case of constructing classical "Shishkin meshes"

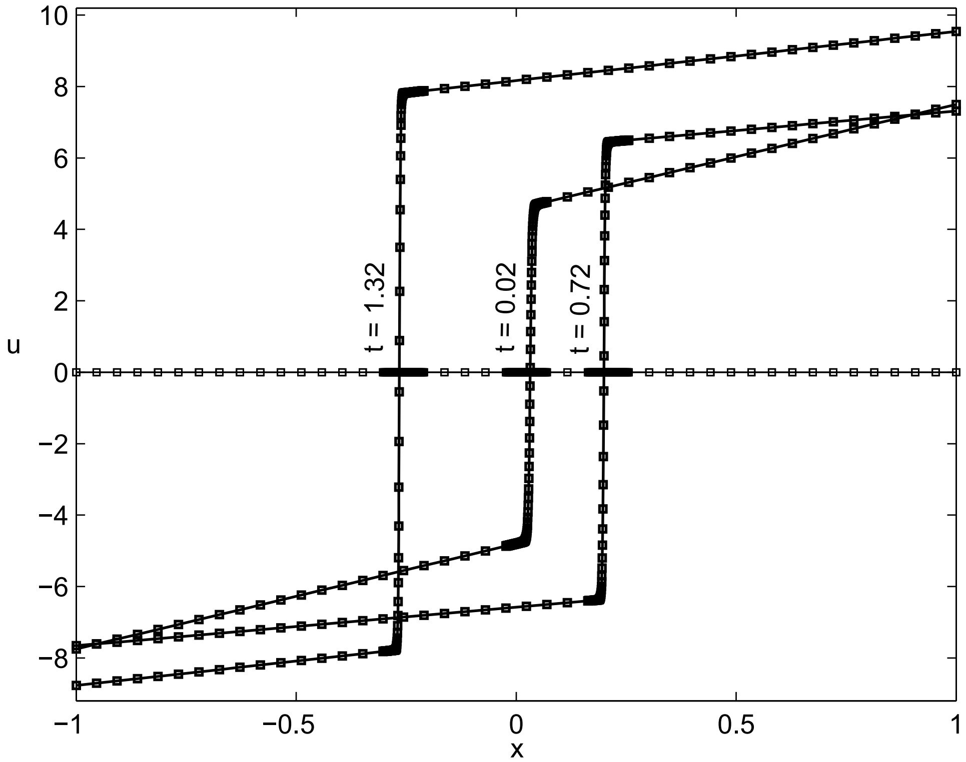

Fig 3. The example of calculation for $\varepsilon = 10^{-2}$. $N_0 = 43$ (has been calculated automatically), $N_{int} = 100$ (control parameter that has been set manually)

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Лукьяненко Д. В., Волков В. Т., Нефедов Н. Н., Реке Л., Шнайдер К., "Аналитико-численный подход для решения сигулярно возмущенных параболических уравнений с использованием динамически адаптированных сеток", *Моделирование и анализ информационых систем*, **23**:3 (2016), 334–341.

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Аннотация. Основной целью данной работы является представление нового аналитико-численного подхода к исследованию сингулярно возмущенных моделей типа реакция-диффузия-адвекция, решения которых содержат движущиеся внутренние переходные слои (фронты). В работе описаны некоторые методы построения динамически адаптированных сеток для эффективного численного решения задач указанного типа. Эти методы основаны на использовании априорной информации о свойствах движущегося фронта, полученной в результате асимптотического анализа. В частности, при построении сетки учитываются априорные асимптотические оценки локализации и скорости фронта, его ширина и структура. Предложенные алгоритмы позволяют существенно снизить затраты вычислительных ресурсов и повысить стабильность численного счета по сравнению с известными классическими подходами.

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Ключевые слова: сингулярно возмущенные параболические уравнения, периодические решения, динамически адаптированные сетки

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