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Error Estimates in Balanced Norms of Finite Element Methods on Shishkin Meshes for Reaction-Diffusion Problems

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Abstract.
Error estimates of finite element methods for reaction-diffusion problems are often realized in the related energy norm. In the singularly perturbed case, however, this norm is not adequate. A different scaling of the H^1 seminorm leads to a balanced norm which reflects the layer behavior correctly.

Keywords: singular perturbation, supercloseness, combination technique, balanced norms

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1. Introduction

We shall examine the finite element method for the numerical solution of the singularly perturbed linear elliptic boundary value problem

$$Lu \equiv -\varepsilon\Delta u + cu = f \qquad \text{in } \Omega = (0,1)\times(0,1) \tag{1a}$$

$$u = 0 \qquad \text{on } \partial\Omega, \tag{1b}$$

where $0 < \varepsilon \ll 1$ is a small positive parameter, $c > 0$ is (for simplicity) a positive constant and f is sufficiently smooth.

The problem has a unique solution $u \in V = H_0^1(\Omega)$ which satisfies in the energy norm

$$\|u\|_\varepsilon := \varepsilon^{1/2}|u|_1 + \|u\|_0 \preceq \|f\|_0. \tag{2}$$

Here we used the following notation: if $A \preceq B$, there exists a (generic) constant C independent of ε (and later also of the mesh used) such that $A \le C\,B$. The error of a finite element approximation $u^N \in V^N \subset V$ satisfies

$$\|u - u^N\|_\varepsilon \preceq \min_{v^N\in V^N} \|u - v^N\|_\varepsilon. \tag{3}$$

When linear or bilinear elements are used on a Shishkin mesh, one can prove for the interpolation error of the Lagrange interpolant $u^I \in V^N$

$$\|u - u^I\|_\varepsilon \preceq \left(\varepsilon^{1/4} N^{-1} \ln N + N^{-2}\right). \tag{4}$$

It follows that the error $u - u^N$ also satisfies such an estimate.

However, the typical boundary layer function $\exp(-x/\varepsilon^{1/2})$ measured in the norm $\|\cdot\|_\varepsilon$ is of order $\mathcal{O}(\varepsilon^{1/4})$. Consequently, error estimates in this norm are less valuable as for convection diffusion equations where the layers are of the structure $\exp(-x/\varepsilon)$. Wherefore we ask the fundamental question:

Is it possible to prove error estimates in the balanced norm

$$\|v\|_b := \varepsilon^{1/4}|v|_1 + \|v\|_0 \quad ? \tag{5}$$

2. The basic error estimate in a balanced norm and some extensions

The mesh Ω^N used is the tensor product of two one-dimensional piecewise uniform Shishkin meshes. I.e., $\Omega^N = \Omega_x \times \Omega_y$, where Ω_x (analogously Ω_y) splits $[0,1]$ into the subintervals $[0, \lambda_x]$, $[\lambda_x, 1-\lambda_x]$ and $[1-\lambda_x, 1]$. The mesh distributes $N/4$ points equidistantly within each of the subintervals $[0, \lambda_x]$, $[1-\lambda_x, 1]$ and the remaining points within the third subinterval. For simplicity, assume

$$\lambda = \lambda_x = \lambda_y = \min\{1/4, \lambda_0\sqrt{\varepsilon/c^*}\ln N\} \quad \text{with } \lambda_0 = 2 \text{ and } c^* < c.$$

Let $V^N \subset H_0^1(\Omega)$ be the space of bilinear finite elements on Ω^N or the space of linear elements over a triangulation obtained from Ω^N by drawing diagonals.

A standard formulation of problem (1) reads: find $u \in V$, such that

$$\varepsilon(\nabla u, \nabla v) + c(u, v) = (f, v) \quad \forall v \in V. \tag{6}$$

By replacing V in (6) with V^N one obtains a standard discretization that yields the FEM-solution u^N. The following estimates for the interpolation error of the Lagrange interpolant hold true:

$$\|u - u^I\|_0 \preceq N^{-2}, \qquad \varepsilon^{1/4}|u - u^I|_1 \preceq N^{-1}\ln N \tag{7}$$

and

$$\|u - u^I\|_{\infty,\Omega_0} \preceq N^{-2}, \quad \|u - u^I\|_{\infty,\Omega\setminus\Omega_0} \preceq (N^{-1}\ln N)^2, \tag{8}$$

here $\Omega_0 = (\lambda_x, 1-\lambda_x) \times (\lambda_y, 1-\lambda_y)$. Let us also introduce $\Omega_f := \Omega \setminus \Omega_0$.

Instead of the Lagrange interpolant we use in our error analysis the L_2 projection $\pi u \in V^N$ from u. Based on

$$u - u^N = u - \pi u + \pi u - u^N$$

we estimate $\xi := \pi u - u^N$:

$$\|\xi\|_\varepsilon^2 \preceq \varepsilon|\nabla\xi|_1^2 + c\,\|\xi\|_0^2 = \varepsilon(\nabla(\pi u - u), \nabla\xi) + c\,(\pi u - u, \xi).$$

Because $(\pi u - u, \xi) = 0$, it follows

$$|\pi u - u^N|_1 \preceq |u - \pi u|_1. \tag{9}$$

If we now could prove a similar estimate as (4) for the error of the L_2 projection, we obtain an estimate in the balanced norm because we have already an estimate for $\|u - u_N\|_0$.

Lemma 1. *The error of the L_2 projection on the Shishkin mesh satisfies*

$$\|u - \pi u\|_\infty \preceq \|u - u^I\|_\infty, \quad \varepsilon^{1/4}|u - \pi u|_1 \preceq N^{-1}(\ln N)^{3/2}. \tag{10}$$

The proof uses the L_∞-stability of the L_2 projection on our mesh [4]. Inverse inequalities are used to move from estimates in W^1_∞ to L_∞, for details see [5].

From Lemma 1 we get

Theorem 1. *The error of the Galerkin finite element method with linear or bilinear elements on a Shishkin mesh satisfies*

$$\|u - u^N\|_b \preceq N^{-1}(\ln N)^{3/2} + N^{-2}. \tag{11}$$

Remark that for Q_k elements with $k > 1$ one can get an analogous result.

It is easy to modify the basic idea to the singularly perturbed semilinear elliptic boundary value problem

$$Lu \equiv -\varepsilon\Delta u + g(\cdot, u) = 0 \qquad \text{in } \Omega = (0,1) \times (0,1) \tag{12a}$$
$$u = 0 \qquad \text{on } \partial\Omega. \tag{12b}$$

We assume that g is sufficiently smooth and $\partial_2 g \geq \mu > 0$.

A standard weak formulation of our semilinear problem reads: find $u \in V$, such that

$$\varepsilon(\nabla u, \nabla v) + (g(\cdot, u), v) = 0 \quad \forall v \in V. \tag{13}$$

By replacing V in (13) with V^N one obtains a standard discretization that yields the FEM-solution u^N. In the error analysis we now use the projection πu defined by

$$(g(\cdot, \pi u), v) = (g(\cdot, u), v) \quad \text{for all } v \in V^N. \tag{14}$$

Our assumption $\partial_2 g \geq \mu > 0$ tells us immediately that πu is well defined, moreover

$$\|u - \pi u\|_0 \preceq \inf_{v \in V^N} \|u - v\|_0. \tag{15}$$

It follows from the definition of our projection

$$|\pi u - u^N|_1 \preceq |u - \pi u|_1. \tag{16}$$

If we now could prove a nice estimate for our projection error in the H^1 seminorm, we would obtain an estimate in the balanced norm because it is easy to estimate $\|u - u_N\|_0$. Based on Taylors formula we can prove

Lemma 2. *The projection defined by (14) is L_∞ stable.*

Similarly as in the linear case we get

Lemma 3. *The projection error of* (14) *on the Shishkin mesh satisfies*

$$\|u - \pi u\|_\infty \preceq \|u - u^I\|_\infty, \quad \varepsilon^{1/4}|u - \pi u|_1 \preceq N^{-1}(\ln N)^{3/2}. \tag{17}$$

Consequently, we get the same error estimate as in Theorem 1 also in the semilinear case.

Next we consider the anisotropic problem

$$-\varepsilon u_{xx} + u_{yy} + cu = f \qquad \text{in } \Omega = (0,1) \times (0,1) \tag{18a}$$
$$u = 0 \qquad \text{on } \partial\Omega. \tag{18b}$$

Now we have only boundary layers at $x = 0$ and $x = 1$. If we want to estimate the error in the balanced norm

$$\|v\|_{b,a} := \varepsilon^{1/4}\|u_x\|_0 + \|u_y\|_0 + \|u\|_0,$$

we start for $\xi := \pi u - u^N$ from

$$\varepsilon\|\xi_x\|_0^2 \le \varepsilon((\pi u - u)_x, \xi_x) + ((\pi u - u)_y, \xi_y) + c\,(\pi u - u, \xi).$$

Now we define in the anisotropic case the projection onto the finite element space by

$$((\pi u - u)_y, \xi_y) + c\,(\pi u - u, \xi) = 0 \quad \forall \xi \in V^N.$$

Consequently it remains to estimate for that projection $\|(\pi u - u)_x\|_0$. But the projection satisfies

$$\pi v = \pi^y(\pi^x\, v),$$

where π^x is the one-dimensional L_2 projection and π^y the one-dimensional Ritz projection (with respect to a non-singularly perturbed operator on a standard mesh), compare [2]. Consequently, the projection is L_∞ stable and we can repeat our basic idea to prove estimates in the balanced norm.

3. Supercloseness and a combination technique

We come back to the linear reaction-diffusion problem

$$Lu \equiv -\varepsilon\Delta u + cu = f \qquad \text{in } \Omega = (0,1) \times (0,1) \tag{19a}$$
$$u = 0 \qquad \text{on } \partial\Omega. \tag{19b}$$

For bilinear elements on the corresponding Shishkin mesh it is well known that we have the supercloseness property (assuming $\lambda_0 \ge 2.5$)

$$\|u^N - u^I\|_\varepsilon \preceq \left(\varepsilon^{1/2}(N^{-1}\ln N)^2 + N^{-2}\right). \tag{20}$$

Can we prove a supercloseness property with respect to the balanced norm?

With $v_N := u^N - \Pi u$ we start from

$$\varepsilon|v_N|_1^2 + c\,\|v_N\|_0^2 \preceq \varepsilon(\nabla(u - \Pi u), \nabla v_N) + c\,(u - \Pi u, v_N).$$

Next we use the decomposition of u into a smooth part S and the layer terms E, i.e., $u = S + E$, decompose also $\Pi u = \Pi S + \Pi E$ and use different projections into our bilinear finite element space for S and E. We choose:

- $\Pi S \in V^N$ satisfies (with given values on the boundary)

$$(\Pi S, v) = (S, v) \quad \forall v \in V_0^N.$$

- ΠE is zero in Ω_0 and the standard bilinear interpolation operator in the fine subdomain with exception of one strip of the width of the fine stepsize in the transition region

With this choice we obtain

$$\varepsilon |v_N|_1^2 + c\,\|v_N\|_0^2 \preceq \varepsilon(\nabla(u - \Pi u), \nabla v_N) + c\,(E - \Pi E, v_N)_{\Omega_f}.$$

In the second term we hope to get some extra power of ε, in the first term we want to apply superconvergence techniques for the estimation of the expression $(\nabla(E - \Pi E), \nabla v_N)$. First let us remark that ΠE satisfies the same estimates as the bilinear interpolant E^I on Ω_f and (based on Lin identities)

$$\varepsilon |(\nabla(E - \Pi E), \nabla v_N)| \preceq N^{-2}\varepsilon^{3/4}|v_N|_1.$$

It is only a technical question to prove that for our modified interpolant using the fact that E is on that strip is as small as we want and that the measure of the strip is small as well.

Summarizing we get the supercloseness result

$$\varepsilon^{1/4}|u^N - \Pi u|_1 \preceq \varepsilon^{1/4}N^{-1} + (N^{-1}\ln N)^2.$$

It is no problem to estimate the L_2 error.

Next we present an application of the supercloseness result to the combination technique. We analyse the version of the combination technique presented in [1].

Writing N for the maximum number of mesh intervals in each coordinate direction, our combination technique simply adds or subtracts solutions that have been computed by the Galerkin FEM on $N \times \sqrt{N}$, $\sqrt{N} \times N$ and $\sqrt{N} \times \sqrt{N}$ meshes. We obtain the same accuracy as on a $N \times N$ mesh with less degrees of freedom. In the following we use the notation of [1].

In the combination technique for bilinear elements we compute a two-scale finite element approximation $u^N_{\hat{N},\hat{N}}$ by

$$u^N_{\hat{N},\hat{N}} := u^N_{N,\hat{N}} + u^N_{\hat{N},N} - u^N_{\hat{N},\hat{N}}.$$

Later we will choose $\hat{N} = \sqrt{N}$. We proved (in our new notation)

$$\|u - u_{NN}\|_b \preceq N^{-1}(\ln N)^{3/2} + N^{-2}. \tag{21}$$

The question is whether or not $u^N_{\hat{N},\hat{N}}$ satisfies a similar estimate (in the case $\hat{N} = \sqrt{N}$).

And indeed our supercloseness result yields finally

$$\|u^N_{\hat{N},\hat{N}} - u_{NN}\|_b \preceq \varepsilon^{1/4}N^{-1/2} + N^{-1}\ln N. \tag{22}$$

That means so far we can only proof the desired estimate for the combination technique if $\varepsilon \preceq N^{-2}$.

4. A direct mixed method

The first balanced error estimate was presented by Lin and Stynes [3] using a first order system least squares (FOSLS) mixed method. For the variables $(u, \bar{q})$ with $-\bar{q} = \nabla u$ and its discretizations on a Shishkin mesh they proved

$$\varepsilon^{1/4}|\bar{q} - \bar{q}^N|_1 + \|u - u^N\|_0 \preceq N^{-1} \ln N. \tag{23}$$

Introducing $\bar{q} = -\nabla u$, a weak formulation of (1) reads:
Find $(u, \bar{q}) \in V \times W$ such that

$$\varepsilon(div\, \bar{q}, w) + c(u, w) = (f, w) \qquad \text{for all } w \in W, \tag{24a}$$
$$\varepsilon(\bar{q}, \bar{v}) - \varepsilon(div\, \bar{v}, u) = 0 \qquad \text{for all } \bar{v} \in V, \tag{24b}$$

with $V = H(div, \Omega)$, $W = L^2(\Omega)$.

For the discretization on a standard rectangular Shishkin mesh we use $(u^N, \bar{q}^N) \in V^N \times W^N$. Here W^N is the space of piecewise constants on our rectangular mesh and V^N the lowest order Raviart-Thomas space RT_0. That means, on each mesh rectangle elements of RT_0 are vectors of the form

$$(span(1, x), span(1, y))^T.$$

Our discrete problem reads: Find $(u^N, \bar{q}^N) \in V^N \times W^N$ such that

$$\varepsilon(div\, \bar{q}^N, w) + c(u^N, w) = (f, w) \qquad \text{for all } w \in W^N, \tag{25a}$$
$$\varepsilon(\bar{q}^N, \bar{v}) - \varepsilon(div\, \bar{v}, u^N) = 0 \qquad \text{for all } \bar{v} \in V^N. \tag{25b}$$

For the error estimation we introduce projections $\Pi : V \mapsto V^N$ and $P : W \mapsto W^N$. As usual, instead of $u - u^N$ and $\bar{q} - \bar{q}^N$ we estimate $Pu - u^N$ and $\Pi\bar{q} - \bar{q}^N$, assuming that we can estimate the projection errors. And indeed we can finally prove

$$\varepsilon^{1/4}\|\Pi\bar{q} - \bar{q}^N\|_0 \preceq N^{-1} \ln N, \qquad \varepsilon^{1/4}\|\nabla u - \bar{q}^N\|_0 \preceq N^{-1} \ln N. \tag{26}$$

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Ханс-Гёрг Роос, "Оценки погрешности в сбалансированных нормах методов конечных элементов на сетках Шишкина для задач реакции-диффузии", *Моделирование и анализ информационых систем*, **23**:3 (2016), 357–363.

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Аннотация. Оценки погрешности методов конечных элементов для задач реакции-диффузии часто производятся в соответствующей энергетической норме. Однако для сингулярно-возмущённого случая такая норма не является адекватной. Перемасштабирование H^1-полунормы приводит к сбалансированной норме, которая правильно отражает поведение переходного слоя.

Ключевые слова: сингулярные возмущения, суперблизость, метод сочетания, сбалансированные нормы

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