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Robust Error Estimation for Singularly Perturbed Fourth Order Problems

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Abstract.
We consider two-dimensional singularly perturbed fourth order problems and estimate on properly constructed layer-adapted errors of a mixed method in the associated energy norms and balanced norms. This paper is a shortened version of [4].

Keywords: singular perturbation, fourth order problem, mixed method, boundary layers, layer-adapted meshes, balanced norms

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Introduction

Let us consider the singularly perturbed plate bending problem given by the fourth-order differential equation

$$\varepsilon^2\Delta^2 u - b\Delta u + (c\cdot\nabla)u + du = f \quad \text{in } \Omega := (0,1)^2, \tag{1a}$$

where $b \geq b_0 > 1$, $d - \frac{1}{2}(\operatorname{div} c + \Delta b) \geq \delta > 0$ and $f \in L^2(\Omega)$ are smooth functions, with the boundary conditions

$$u = \frac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma := \partial\Omega. \tag{1b}$$

The solution of this problem lies in H_0^2 which means, a conforming finite element discretisation requires C^1-elements. They are not very popular in 2d or 3d, which leads to the widely usage of mixed or non-conforming methods. In this paper we want to study mixed finite element methods of order p. For non-singularly perturbed problems ($\varepsilon = 1$) and p-th order finite-element approximation for u and $w = \Delta u$, the classical error estimate

$$\|u - u_h\|_1 + h\|w - w_h\|_0 \leq Ch^p\|u\|_{p+1} \tag{2}$$

for the discrete solutions u_h and w_h on a standard shape-regular mesh with $p \geq 2$, is known, see [3, 11].

Here we want to consider the singularly perturbed case and estimation in energy and balanced norms. We use standard notation for Sobolev spaces, where $\|\cdot\|_0$ is the L^2-norm, $|\cdot|_k$ the seminorm in H^k and $\|\cdot\|_k$ the full H^k-norm. Furthermore, we denote by $\langle u, v\rangle_D$ the L^2-scalar product over a domain $D \subset \Omega$, a subscript we drop if $D = \Omega$.

1. Solution decomposition and meshes

For our numerical analysis to work we assume a decomposition of the solution u of problem (1a)+(1b) into a smooth part, boundary layers and corner layers:

$$u = S + \sum_{k\in\mathcal{I}} E_k, \quad \text{where} \quad \mathcal{I} = \{1, 2, 3, 4, 12, 23, 34, 41\}.$$

Here S stands for the smooth part, E_k with $k = 1, 2, 3, 4$ for a boundary layer and E_k with $k = 12, 23, 34, 41$ for a corner layer. More precisely, we assume

$$\|\partial_x^i \partial_y^j S\|_0 \leq C, \qquad |\partial_x^i \partial_y^j E_1(x, y)| \leq C\varepsilon^{1-i} \mathrm{e}^{-x/\varepsilon},$$
$$|\partial_x^i \partial_y^j E_2(x, y)| \leq C\varepsilon^{1-j} \mathrm{e}^{-y/\varepsilon}, \qquad |\partial_x^i \partial_y^j E_{12}(x, y)| \leq C\varepsilon^{1-i-j} \mathrm{e}^{-x/\varepsilon} \mathrm{e}^{-y/\varepsilon},$$

and similarly for the other components of the decomposition. These assumptions are reasonable, see e.g in 1d in [12] or for smooth domains in [2, §12.4.3].

Using the information on the layers we generate a layer-adapted Shishkin mesh [15]. With the transition points $\lambda = \sigma\varepsilon \ln N < \frac{1}{4}$ the interval $[0, 1]$ is now partitioned with a piecewise equidistant mesh, that is constructed by equidistantly dividing $[0, \lambda]$ into $N/4$ subintervals, $[\lambda, 1 - \lambda]$ into $N/2$ and $[1 - \lambda, 1]$ into $N/4$ subintervals again. The tensor product of two such one-dimensional meshes gives the Shishkin mesh.

With above assumption on the solution decompositions we have $|E_1(\lambda, y)| \leq C\varepsilon N^{-\sigma}$. These layers are therefore *weak layers* as their influence vanishes with decreasing ε in a pointwise sense. Note also that the small and the large meshwidths satisfy

$$h = \frac{4\lambda}{N} \leq C\varepsilon N^{-1} \ln N \quad \text{and} \quad H = 2\frac{1 - 2\lambda}{N} \leq CN^{-1}.$$

2. Numerical method and analysis

Using $w = \varepsilon\Delta u \in H^2(\Omega)$, we rewrite the fourth-order problem as a system and obtain a weak formulation:

Find $(u, w) \in H_0^1(\Omega) \times H^1(\Omega)$ such that for all $(\varphi, \psi) \in H^1(\Omega) \times H_0^1(\Omega)$

$$\varepsilon\langle\nabla u, \nabla\varphi\rangle + \langle w, \varphi\rangle = 0,$$
$$\langle b\nabla u, \nabla\psi\rangle + \langle(\mathbf{c} \cdot \nabla)u + du, \psi\rangle - \varepsilon\langle\nabla w, \nabla\psi\rangle = \langle f, \psi\rangle,$$

where $\mathbf{c} = c + \nabla b$. The associated bilinear form is given by $a : (H_0^1 \times H^1)^2 \mapsto \mathbb{R}$ with

$$a((u, w), (\psi, \varphi)) = \varepsilon\langle\nabla u, \nabla\varphi\rangle + \langle w, \varphi\rangle + \langle b\nabla u, \nabla\psi\rangle + \langle\mathbf{c} \cdot \nabla u + du, \psi\rangle - \varepsilon\langle\nabla w, \nabla\psi\rangle.$$

and a corresponding energy norm by

$$|||(u,w)|||^2 := \|w\|_0^2 + b_0\|\nabla u\|_0^2 + \delta\|u\|_0^2.$$

By a direct calculation we have coercivity of the bilinear form w.r.t. the energy norm

$$a((u,w),(u,w)) \geq |||(u,w)|||^2 .$$

Therefore, above mixed formulation has a unique solution. Now let us define the discrete space on a rectangularly divided mesh T_N. We use

$$V := \{v \in H^1(\Omega) : v|_\tau \in \mathcal{Q}_p(\tau)\,\forall\tau \in T_N\}, \quad V_0 := V \cap H_0^1(\Omega).$$

Here $\mathcal{Q}_p(\tau)$ is the polynomial space on τ, with polynomial degrees at most p in each coordinate direction. The discrete problem now reads: Find $(u_h, w_h) \in V_0 \times V$ such that

$$a((u_h,w_h),(\psi,\varphi)) = \langle f,\psi\rangle \quad \text{for all } \varphi \in V,\ \psi \in V_0. \tag{3}$$

2.1. Estimation in the energy norm

The analysis of our method works, as usual, with the help of suitable interpolation operators and their error estimates. Let us define the interpolation operators $I : C(\Omega) \to V_0(\Omega)$ and $J : C(\Omega) \to V(\Omega)$ in the sense of [5, p. 108] and [8]. For these interpolation operators hold the anisotropic interpolation-error estimates by [1, 10]. Using these, standard techniques and the definition of the mesh, see i.e. [14, Section III.3.5], we obtain the following interpolation-error estimates.

Lemma 1. *We have for* $\sigma \geq p+1$

$$\|u - Iu\|_0 \leq C(N^{-1}\ln N)^{p+1}, \qquad \|\nabla(u-Iu)\|_0 \leq C(N^{-1}\ln N)^{p},$$
$$\|w - Jw\|_0 \leq C(N^{-1}\ln N)^{p+1}, \qquad \|\nabla(w-Jw)\|_0 \leq C\varepsilon^{-1/2}(N^{-1}\ln N)^{p}.$$

We can also use supercloseness estimates based on integral identities from [7, 6, 17, 16]. They yield on each cell τ

$$|\langle (Iv-v)_x, \chi_x\rangle_\tau| \leq Ck_\tau^{p+1}\|v_{xy^{p+1}}\|_{0,\tau}\|\chi_x\|_{0,\tau}. \tag{4a}$$

Let us now consider a rectangular domain $T = \bigcup\{\tau\}$ with $\ell_{1,T}$ and $\ell_{2,T}$ being its left and right boundary. With inverse and Holder inequalities we conclude also the estimates

$$\begin{aligned}|\langle (Iv-v)_x, \chi_x\rangle_T| \leq{}& C\sum_{\tau\subset T} k_\tau^{p+1}\left(\frac{k_\tau}{h_\tau}\|v_{xy^{p+2}}\|_{0,\tau} + \|v_{x^2y^{p+1}}\|_{0,\tau}\right)\|\chi\|_{0,\tau}\\ &+ C\sum_{i=1}^{2}\sum_{\substack{\tau\subset T\\ \tau\cap\ell_{i,T}\neq\emptyset}} k_\tau^{p+1}\left(\frac{k_\tau}{h_\tau}\right)^{1/2}\|v_{xy^{p+1}}\|_{L^\infty(\partial\tau\cap\ell_{i,T})}\|\chi\|_{0,\tau}.\end{aligned} \tag{4b}$$

If $v_x = 0$ or $\chi = 0$ on $\ell_{i,T}$ for $i = 1$ or $i = 2$, then the sum containing i in (4b) can be omitted. The proof of the following theorem is based purely on interpolation-error estimates, supercloseness estimates and standard techniques for singularly perturbed problems.

Theorem 2. *Let $(u_h, w_h) \in V_0 \times V$. Then it holds for the discrete error on a Shishkin mesh with $\sigma \geq p+2$ the supercloseness result*

$$|||(Iu - u_h, Jw - w_h)||| \leq C(N^{-1} \ln N)^{p+1}.$$

Having these discrete-error and interpolation-error estimates, it is easy to conclude the following error estimates.

Theorem 3. *On a Shishkin mesh with $\sigma \geq p+2$ holds for the exact solution (u, w) and the discrete solution $(u_h, w_h) \in V_0 \times V$*

$$|||(u - u_h, w - w_h)||| \leq C(N^{-1} \ln N)^{p}.$$

2.2. Estimation in a balanced norm

For a typical layer function E_1 and a non-layer function S the energy-norm yields

$$|||(E_1, \varepsilon\Delta E_1)|||^2 = \varepsilon^2 \|\Delta E_1\|_0^2 + b_0 \|\nabla E_1\|_0^2 + \delta \|E_1\|_0^2 \leq C\varepsilon \text{ and } |||(S, \varepsilon\Delta S)|||^2 \leq C$$

and the the layer is not seen for $\varepsilon \to 0$ in the energy norm. Introducing a balanced norm

$$|||(u, w)|||_b^2 := \varepsilon^{-1} \|w\|_0^2 + b_0 \|\nabla u\|_0^2 + \delta \|u\|_0^2$$

with

$$|||(E_1, \varepsilon\Delta E_1)|||_b^2 = \varepsilon \|\Delta E_1\|_0^2 + b_0 \|\nabla E_1\|_0^2 + \delta \|E_1\|_0^2 \leq C \quad \text{and} \quad |||(S, \varepsilon\Delta S)|||_b^2 \leq C,$$

the layer is seen in this balanced norm. Unfortunately, our method is not coercive with respect to this norm. In order to prove error estimates we have to combine some more ideas. In [9] a special interpolation operator is constructed which uses different interpolants for a decomposition of $w = \varepsilon\Delta u$ into smooth and layer components and is for layer components zero in the subdomain where the layer components are small enough. In [13] the idea of using suitable projections in estimating balanced norms is introduced. For this purpose we define the Ritz-projection $\pi u \in V_0$ by

$$\langle b\nabla(\pi u - u), \nabla\psi\rangle + \langle \mathbf{c} \cdot \nabla(\pi u - u), \psi\rangle + \langle d(\pi u - u), \psi\rangle = 0$$

for all $\psi \in V_0$. Then we have for $\psi = \pi u - u_h \in V_0$ and $\varphi = \bar{J}w - w_h \in V$.

$$\big|||(\pi u - u_h, \bar{J}w - w_h)\big|||^2 \leq \varepsilon\langle\nabla(\pi u - u), \nabla\varphi\rangle + \langle \bar{J}w - w, \varphi\rangle - \varepsilon\langle\nabla(\bar{J}w - w), \nabla\psi\rangle.$$

Using these ideas we can prove error estimates in the stronger balanced norm.

Theorem 4. *It holds for the error of the discrete solution $(u_h, w_h) \in V_0 \times V$ in the balanced norm*

$$|||(u - u_h, w - w_h)|||_b \leq C(N^{-1} \ln N)^{p-1}.$$

Remark 5. *Comparing the results of the Theorems 3 and 4 we observe a reduction of the convergence order by one when measuring in the stronger norm. Numerically, there is no reduction visible.*

Moreover, combining Theorems 3 and 4 we obtain

$$\|u - u_h\|_1 + (N^{-1} \ln N)\varepsilon^{-1/2} \|w - w_h\|_0 \leq C(N^{-1} \ln N)^{p}$$

which is the corresponding uniform result on Shishkin meshes to the classical estimate (2) for $\varepsilon = 1$.

2.3. Further problems

The techniques used for the error estimates presented here can also be used for related problems. One of these examples is

$$\begin{aligned} \varepsilon^2\Delta^2\tilde{u} - b\Delta\tilde{u} + (c\cdot\nabla)\tilde{u} + d\tilde{u} = f \quad &\text{in } \Omega := (0,1)^2, \\ \tilde{u} = \Delta\tilde{u} = 0 \quad &\text{on } \Gamma. \end{aligned}$$

These boundary conditions introduce even weaker boundary layers, but above analysis can be adapted for this case too, and the same results in the energy norm hold. This norm is again not balanced, but can be made stronger by properly defining the weights in its definition. In this stronger norm an error estimate of the same order as in the energy norm holds in the case of constant b.

The extension to problems of type

$$\varepsilon^2\Delta^2\hat{u} + d\hat{u} = f \quad \text{in } \Omega := (0,1)^2,$$

with either of the boundary conditions considered above, can also be done. Here the boundary layers are stronger than the ones considered so far. Still the analysis can be applied.

Further information, the full proofs and numerical results can be found in [4].

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Аннотация. Рассматриваются двумерные сингулярно возмущённые задачи четвертого порядка и оцениваются должным образом построенные адаптированные к слою погрешности смешанного метода в соответствующих энергетических нормах и сбалансированных нормах. Данная работа является сокращенной версией [4].

Ключевые слова: сингулярные возмущения, задача четвертого порядка, смешанный метод, пограничные слои, адаптированные к слою сетки, сбалансированные нормы

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