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A Caputo Two-Point Boundary Value Problem: Existence, Uniqueness and Regularity of a Solution

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Abstract.
A two-point boundary value problem on the interval $[0,1]$ is considered, where the highest-order derivative is a Caputo fractional derivative of order $2-\delta$ with $0<\delta<1$. A necessary and sufficient condition for existence and uniqueness of a solution u is derived. For this solution the derivative u' is absolutely continuous on $[0,1]$. It is shown that if one assumes more regularity — that u lies in $C^2[0,1]$ — then this places a subtle restriction on the data of the problem.

Keywords: fractional derivative, boundary value problem, existence, uniqueness, regularity

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Introduction

Fractional derivatives are very fashionable at present: they are used in many recent models to give results that seem to be unattainable by classical integer-order derivatives. Consequently there is a huge amount of current research activity in the area of numerical methods for the solution of differential equations that involve fractional-order derivatives. Unfortunately, many papers analysing numerical methods for fractional-derivative problems neglect to discuss existence, uniqueness and regularity of the the solution to the problem they are solving — these fundamental and crucial properties are simply assumed to be true!

In the present paper, which is partly based on [5], we consider a Caputo two-point boundary value that is defined in Section 1.. This problem models superdiffusion of particle motion when convection is present; see [3, Section 1] and its references. In Section 2. we shall derive a necessary and sufficient condition for existence and uniqueness of a solution to this problem in a certain space of functions that lies between $C^1[0,1]$ and $C^2[0,1]$. Then in Section 3. we show that if one assumes that u lies in $C^2[0,1]$ — i.e., one assumes more regularity of the solution — then this places a subtle restriction on the data of the problem.

Notation. All functions are real valued. $C(I)$ comprises those functions that are continuous on an interval I, and $C^k(I)$ denotes the space of functions defined on I whose derivatives up to order k lie in $C(I)$, for $k = 1, 2, \dots$ We follow the convention that $C^0(I) = C(I)$. Denote by $L_1[0,1]$ the standard Lebesgue space of integrable functions defined almost everywhere on $[0,1]$.

1. The Caputo two-point boundary value problem

The following definitions are needed to describe our boundary value problem.

For $r \in \mathbb{R}$ with $r > 0$, and all $g \in L_1[0,1]$, the *Riemann-Liouville fractional integral operator* J^r of order r is defined by

$$(J^r g)(x) = \left[\frac{1}{\Gamma(r)} \int_{t=0}^{x} (x-t)^{r-1} g(t)\, dt\right] \quad \text{for} \quad 0 \le x \le 1. \tag{1}$$

Let the parameter δ satisfy $0 < \delta < 1$. Let $g \in C^1[0,1]$ with g' absolutely continuous on $[0,1]$. Then the *Caputo fractional derivative* $D_*^{2-\delta} g$ of order $2-\delta$ is defined for almost all $x \in (0,1)$ (see, e.g., [4, Theorem 2.1]) by

$$D_*^{2-\delta} g(x) := (J^\delta g'')(x) = \frac{1}{\Gamma(\delta)} \int_{t=0}^{x} (x-t)^{\delta-1} g''(t)\, dt \quad \text{for} \quad 0 < x \le 1. \tag{2}$$

Since the integral in $D_*^{2-\delta} g(x)$ is associated with the point $x = 0$, many authors write instead $D_{*0}^{2-\delta} g(x)$, but for simplicity of notation we omit the subscript 0.

We shall consider the *two-point boundary value problem*

$$-D_*^{2-\delta} u(x) + b(x) u'(x) + c(x) u(x) = f(x) \quad \text{for } x \in (0,1), \tag{3a}$$

subject to the boundary conditions

$$u(0) - \alpha_0 u'(0) = \gamma_0, \tag{3b}$$
$$u(1) + \alpha_1 u'(1) = \gamma_1, \tag{3c}$$

where the constants $\alpha_0, \alpha_1, \gamma_0, \gamma_1$ and the functions b, c and f are given. We assume that $b, c, f \in C^q[0,1]$ for some integer $q \ge 1$.

Remark 1. *If one also assumes that $c \ge 0$ on $[0,1]$, $\alpha_0 \ge 1/(1-\delta)$ and $\alpha_1 \ge 0$, then (3) satisfies a comparison/maximum principle, from which existence and uniqueness of the solution u of (3) follows; see [7]. But if the Robin boundary condition at $x = 0$ is replaced by a Dirichlet boundary condition, then the comparison/maximum principle may no longer be true: a counterexample is given in [7, Example 2.4].*

A more general class of boundary value problems is considered in [6]. Numerical methods for the solution of (3) are presented and analysed rigorously in, for instance, [3, 5, 6, 7].

The present paper will discuss some theoretical aspects of (3): existence, uniqueness and regularity of solutions. Existence and uniqueness of a solution using the space

$C^{q,\delta}(0,1]$ defined below was proved in [5] by means of a reformulation in terms of Volterra integral equations of the second kind, under the additional hypotheses that

$$c \geq 0, \ \alpha_0 \geq 0 \quad \text{and} \quad \alpha_1 \geq 0. \tag{4}$$

We shall use the same Volterra reformulation here but interpret its conclusions in a more general way that yields conditions on the data that are necessary and sufficient for existence and uniqueness of a solution to (3).

When discussing solutions of (3), the following setting is natural [1, 8]. Let $C^{q,\delta}(0,1]$ be the space of all functions $y \in C[0,1] \cap C^q(0,1]$ such that

$$\|y\|_{q,\delta} := \sup_{0<x\leq 1} |y(x)| + \sum_{k=1}^{q} \sup_{0<x\leq 1} \left[x^{k-(1-\delta)} \left|y^{(k)}(x)\right|\right] < \infty.$$

That is, one has $|y(x)| \leq C$ and $|y^{(k)}(x)| \leq Cx^{(1-\delta)-k}$ for $k = 1, \dots, q$. By [8], $C^{q,\delta}(0,1]$ is a Banach space. Note that $C^q[0,1] \subset C^{q,\delta}(0,1]$.

Define the space of functions

$$C_1^{q,\delta}(0,1] := \left\{ y \in C^1[0,1] \cap C^{q+1}(0,1] : y' \in C^{q,\delta}(0,1] \right\}.$$

We are interested only in those solutions u of (3) that lie in $C_1^{q,\delta}(0,1]$. This is a reasonable class of candidates for solutions of (3), since then $D_*^{2-\delta}u$ is defined everywhere in $(0,1]$ by Lemma 1 below, and as we shall see in Section 3., imposing more regularity on u'' by requiring $u \in C^2[0,1]$ would lead to certain difficulties.

Lemma 1. *Let $y \in C_1^{q,\delta}(0,1]$. Then $D_*^{2-\delta}y(x)$ is defined for all $x \in (0,1]$.*

Proof. Let $x \in (0,1]$. Then $D_*^{2-\delta}y(x) = (1/\Gamma(\delta)) \int_{t=0}^{x} (x-t)^{\delta-1} y''(t)\, dt$, provided this integral exists. Invoking the hypothesis that $y \in C_1^{q,\delta}(0,1]$, one has

$$\frac{1}{\Gamma(\delta)} \int_{t=0}^{x} \left|(x-t)^{\delta-1} y''(t)\right| dt \leq \frac{C}{\Gamma(\delta)} \int_{t=0}^{x} (x-t)^{\delta-1} t^{-\delta}\, dt = C\,\Gamma(1-\delta)$$

by a standard formula for Euler's Beta function [2, Theorem D.6]. Hence the integral exists in the Lebesgue sense, i.e., $D_*^{2-\delta}y(x)$ is defined. □

Example 1. *Consider the simple problem $D_*^{2-\delta}u = \Gamma(3-\delta)$ on $(0,1)$, $u(0) = 0$, $u(1) = 1$. From [2, pages 55 and 193] it is easy to see that the unique solution u of this problem is $u(x) = x^{2-\delta}$. Hence $u \in C_1^{m,\delta}(0,1]$ for any positive integer m, but $u \notin C^2[0,1]$.*

The regularity of the solution of Example 1 is typical of solutions to the general boundary value problem (3).

2. Existence and uniqueness of a solution

Define the Volterra operator L by

$$Lz(x) = z(x) - \frac{1}{\Gamma(1-\delta)} \int_{t=0}^{x} (x-t)^{-\delta} \left[b(t)z(t) + c(t) \int_0^t z(s)\, ds \right] dt \quad \text{for } 0 \leq x \leq 1.$$

It is shown in the proof of [5, Lemma 2.1] that $L : C^{q,\delta}(0,1] \to C^{q,\delta}(0,1]$ is a compact operator.

Consider now two Volterra integral equations of the second kind: for $0 \le x \le 1$,

$$Lv(x) = \frac{1}{\Gamma(1-\delta)} \int_{t=0}^{x} (x-t)^{-\delta}[b(t) + (t+\alpha_0)c(t)]\, dt \tag{5}$$

and

$$Lw(x) = \frac{1}{\Gamma(1-\delta)} \int_{t=0}^{x} (x-t)^{-\delta}[\gamma_0 c(t) - f(t)]\, dt. \tag{6}$$

From [5, Lemma 4.1], the solutions v and w are well defined and lie in $C^{q,\delta}(0,1]$.

Theorem 1 (Existence and uniqueness of a solution to (3)). *Set*

$$\theta = \alpha_0 + \alpha_1[1 + v(1)] + \int_0^1 [1+v].$$

1. *If $\theta \neq 0$, then (3) has a unique solution*

$$u(x) = \gamma_0 + \mu\alpha_0 + \int_{t=0}^{x} [\mu(1+v(t)) + w(t)]\ dt$$

with $u \in C_1^{q,\delta}(0,1]$, where

$$\mu = \frac{\gamma_1 - \gamma_0 - \alpha_1 w(1) - \int_0^1 w}{\theta}. \tag{7}$$

2. *If $\theta = 0$, then (3) has either no solution or infinitely many solutions in $C_1^{q,\delta}(0,1]$.*

Proof. The analysis of [5] shows that for any $\mu \in \mathbb{R}$ the function

$$u(x) = u(0) + \mu x + \int_0^x (\mu v + w)(t)\, dt \tag{8}$$

lies in $C_1^{q,\delta}(0,1]$ and will satisfy the differential equation (3a) and the boundary condition (3b); it is also shown in [5] that if a function $u \in C_1^{q,\delta}(0,1]$ satisfies (3a) and (3b), then u satisfies (8). Thus it remains only to choose μ in (8) such that u satisfies the boundary condition (3c): $u(1) + \alpha_1 u'(1) = \gamma_1$.

Using (8) and eliminating $u(0)$ by means of (3b), one has

$$\begin{aligned} u(1) + \alpha_1 u'(1) &= u(0) + \mu + \alpha_1[\mu + \mu v(1) + w(1)] + \int_0^1 (\mu v + w) \\ &= \gamma_0 + \alpha_0\mu + u(0) + \mu + \alpha_1[\mu + \mu v(1) + w(1)] + \int_0^1 (\mu v + w) \\ &= \gamma_0 + \mu\theta + \alpha_1 w(1) + \int_0^1 w. \end{aligned}$$

If $\theta \neq 0$, then the unique choice of μ given by (7) yields $u(1) + \alpha_1 u'(1) = \gamma_1$ and the solution of (3) is then specified by (8).

If $\theta = 0$, there are two possibilities: if $\gamma_0 + \alpha_1 w(1) + \int_0^1 w \neq \gamma_1$ then the boundary condition (3c) cannot be satisfied and (3) has no solution, while if $\gamma_0 + \alpha_1 w(1) + \int_0^1 w = \gamma_1$, then the boundary condition (3c) is satisfied for any choice of μ and we have infinitely many solutions given by (8) where $\mu \in \mathbb{R}$ is arbitrary. □

In [5, Theorem 4.1] it was shown that when (4) is satisfied, one then has $\theta > 0$ and consequently (3) has a unique solution, but the more general situation described in Theorem 1 was not discussed.

3. Effect of assuming that $u \in C^2[0, 1]$

In Sections 1. and 2., solutions of (3) lying in the space $C_1^{q,\delta}(0, 1]$ were considered. These solutions are smooth on $(0, 1]$ but typically much less smooth on the *closed* interval $[0, 1]$. The present section examines the effect of assuming that the solution u lies not just in $C_1^{q,\delta}(0, 1]$ but in the space $C^2[0, 1]$ for which u'' is bounded on $[0, 1]$. We show that with this assumption, the class of problems under consideration is restricted more severely than one would expect.

Higher regularity of solutions on the *closed* interval $[0, 1]$ is commonly assumed in numerical analyses of fractional-derivative problems, but many researchers seem unaware of the consequences of this assumption. We describe here what $u \in C^2[0, 1]$ implies for our problem (3); our results can easily be generalised to Caputo differential equations (boundary value problems and initial-value problems) of any order.

The crucial observation is the following result (see, e.g., [2, Lemma 3.11]), whose short elementary proof we include for completeness.

Lemma 2. *Let $g \in C^2[0, 1]$. Then*

$$\lim_{x \to 0^+} D_*^{2-\delta} g(x) = 0.$$

Proof. For any $x \in (0, 1)$,

$$D_*^{2-\delta} g(x) = \frac{1}{\Gamma(\delta)} \int_{t=0}^{x} (x - t)^{\delta - 1} g''(t)\, dt.$$

But $g \in C^2[0, 1]$ implies that $|g''(t)| \leq C$ for $0 \leq t \leq 1$ and some constant C. Hence

$$\left| D_*^{2-\delta} g(x) \right| \leq \frac{C}{\Gamma(\delta)} \int_{t=0}^{x} (x - t)^{\delta - 1}\, dt = \frac{C x^{\delta}}{\Gamma(\delta + 1)} \to 0 \text{ as } x \to 0^+.$$

□

Remark 2. *The converse of Lemma 2 is false. For suppose $g(x) = x^{2-\beta}$ with $0 < \beta < \delta$. Then $g \in C_1^{q,\delta}(0, 1]$ but $g \notin C^2[0, 1]$, and*

$$D_*^{2-\delta} g(x) = \frac{(2 - \beta)(1 - \beta)}{\Gamma(\delta)} \int_{t=0}^{x} (x - t)^{\delta - 1} t^{-\beta}\, dt = \frac{\Gamma(3 - \beta)}{\Gamma(1 + \delta - \beta)} x^{\delta - \beta} \to 0 \text{ as } x \to 0^+,$$

where we used the standard formula for Euler's Beta function [2, Theorem D.6] to evaluate the integral.

Remark 3. *By imitating the calculation of Remark 2, one can replace the hypothesis* $g \in C^2[0,1]$ *of Lemma 2 by the weaker assumption that* $g \in C_1^{2,\beta}(0,1]$ *for some* $\beta \in (0,\delta)$.

Assume now that (3) has a solution $u \in C^2[0,1]$. Then by Lemma 2 we can apply $\lim_{x\to 0^+}$ to (3a), obtaining

$$b(0)u'(0) + c(0)u(0) = f(0). \tag{9}$$

One also has the boundary condition (3b); combining this with (9) yields

$$f(0) = [b(0) + \alpha_0 c(0)]u'(0) + \gamma_0 c(0). \tag{10}$$

Assume that u is the unique solution of (3), i.e., assume that $\theta \neq 0$ in Theorem 1. Then the value of $u'(0)$ is given by (7). Thus f must satisfy the equation

$$f(0) = \frac{[b(0) + \alpha_0 c(0)][\gamma_1 - \gamma_0 - \alpha_1 w(1) - \int_0^1 w]}{\alpha_0 + \alpha_1[1 + v(1)] + \int_0^1 [1+v]} + \gamma_0 c(0). \tag{11}$$

As w depends on f by (6), the necessary condition (11) places a difficult-to-verify restriction on f that is completely unnatural, and is due entirely to the arbitrary assumption that $u \in C^2[0,1]$.

Remark 4. *In the special case where* $b \equiv 0$ *and* $\alpha_0 = 0$, *the problem* (3) *becomes*

$$-D_*^{2-\delta}u + cu = f \text{ on } (0,1), \text{ with } u(0) = \gamma_0,\ u(1) + \alpha_1 u'(1) = \gamma_1.$$

If $u \in C^2[0,1]$ *here, we can work directly from Lemma 2 without appealing to [5]: taking the limit of the differential equation as* $x \to 0^+$ *shows that*

$$f(0) = c(0)u(0) = c(0)\gamma_0$$

is a necessary condition for a solution u *in* $C^2[0,1]$.

The analysis of this section shows that making excessive regularity assumptions on the solution to a fractional-derivative is not only unjustified (recall Example 1) but also restricts the class of problems under consideration by imposing a condition (11) on the data that may be difficult to check in any concrete example.

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Аннотация. Рассматривается двухточечная краевая задача на промежутке $[0, 1]$, в которой старшая производная является дробной производной Капуто порядка $2 - \delta$ при $0 < \delta < 1$. Получено необходимое и достаточное условие существования и единственности решения u. Производная u' этого решения оказывается абсолютно непрерывной на $[0, 1]$. Показано, что предположение о большей регулярности — что u принадлежит $C^2[0, 1]$ — накладывает довольно тонкое ограничение на данные задачи.

Ключевые слова: дробная производная, краевая задача, существование, единственность, регулярность

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