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Fibred Product of Commutative Algebras: Generators and Relations

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Abstract. The method of direct computation of the universal (fibred) product in the category of commutative associative algebras of finite type with unity over a field is given and proven. The field of coefficients is not supposed to be algebraically closed and can be of any characteristic. Formation of fibred product of commutative associative algebras is an algebraic counterpart of gluing algebraic schemes by means of some equivalence relation in algebraic geometry. If initial algebras are finite-dimensional vector spaces, the dimension of their product obeys a Grassmann-like formula. A finite-dimensional case means geometrically the strict version of adding two collections of points containing a common part.

The method involves description of algebras by generators and relations on input and returns similar description of the product algebra. It is "ready-to-eat" even for computer realization. The product algebra is well-defined: taking other descriptions of the same algebras leads to isomorphic product algebra. Also it is proven that the product algebra enjoys universal property, i.e. it is indeed a fibred product. The input data are a triple of algebras and a pair of homomorphisms $A_1 \stackrel{f_1}{\to} A_0 \stackrel{f_2}{\leftarrow} A_2$. Algebras and homomorphisms can be described in an arbitrary way. We prove that for computing the fibred product it is enough to restrict to the case when $f_i, i = 1, 2$ are surjective and describe how to reduce to the surjective case. Also the way of choosing generators and relations for input algebras is considered.

Paper is published in the author's wording.

Keywords: commutative algebras over a field, affine Grothendieck' schemes, universal product, amalgamated sum

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Introduction

0.1. Motivation and General Problem

For a motivating example consider affine plane $\mathbb{A}^2$ over a field k and two non-proportional irreducible polynomials g_1, g_2 in two variables x, y over the same field. Let these polynomials have zero sets $Z(g_1) \subset \mathbb{A}^2 \supset Z(g_2)$ and a set of common zeros $Z(g_1, g_2) = Z(g_1) \cap Z(g_2)$. If k is algebraically closed $Z(g_i)$ are irreducible curves for any $g_i, i = 1, 2$, and $Z(g_1, g_2)$ is a discrete collection of points. A union of curves $Z(g_1) \cup Z(g_2)$ is given by zeros of the product polynomial $g_1 g_2$, i.e. $Z(g_1) \cup Z(g_2) = Z(g_1 g_2)$ and provides a simple example of amalgamated sum $Z(g_1) \coprod_{Z(g_1,g_2)} Z(g_2)$ of algebraic schemes $Z(g_i), i = 1, 2$ (precise definition will be given below). Transferring to algebraic counterpart one has k-algebras $A_i = k[x, y]/(g_i), i = 1, 2$ corresponding to curves $Z(g_i), i = 1, 2$ respectively, and the k-algebra of intersection locus $A_0 = k[x, y]/(g_1, g_2)$. There are obvious k-algebra homomorphisms $f_i : A_i \twoheadrightarrow A_0$. Formation of the algebra $k[x, y]/(g_1 g_2)$ for the union of two components provides an example of universal (fibred) product of k-algebras $A_1 \times_{A_0} A_2$.

For our purposes it is enough to restrict by affine algebraic k-schemes of finite type, i.e. prime spectra (sets of prime ideals with natural topology and collection of local rings forming a structure sheaf on the spectrum) of commutative associative k-algebras of finite type with unity (for complete theory cf. [1, ch.2]). We focus on algebraic side and operate exclusively with underlying algebras.

We describe general *quotient problem* in the category of algebraic schemes over a field k. The problem includes following ingredients:

- a scheme X;
- a subscheme $R \subset X \times X$;
- morphisms $p_i : R \subset X \times X \to X$ defined as composites of the immersion with the projection on ith factor.

The subscheme $R \subset X \times X$ is said to be an *equivalence relation*; this means that it satisfies three requirements as follows:

1. R is *reflexive*, i.e. $R \supset \operatorname{diag}(X)$ where $\operatorname{diag}(X)$ is an image of the diagonal immersion $\operatorname{diag} : X \hookrightarrow X \times X$;
2. R is *symmetric*, i.e. the immersion $R \subset X \times X$ is stable under the involution intertwining first and second factors of the product $X \times X$;
3. R is *transitive*, i.e. $\mathrm{pr}_{13}(R \times X \cap X \times R) \subset R$ where the intersection is taken in $X \times X \times X$. Here $\mathrm{pr}_{13} : X \times X \times X \to X \times X$ is the projection to the product of the 1st and 3rd factors.

The question is [2, ch. 1, 4.3] to construct (if possible) the *universal quotient* X/R. This is an object fitting into the commutative diagram

$$\begin{tikzcd} R \arrow[r, "p_2"] \arrow[d, "p_1"'] & X \arrow[d] \\ X \arrow[r] & X/R \end{tikzcd}$$

such that for any other commutative square

$$\begin{tikzcd} R \arrow[r, "p_2"] \arrow[d, "p_1"'] & X \arrow[d] \\ X \arrow[r] & T \end{tikzcd}$$

there is a unique morphism $\tau : X/R \to T$ (in the appropriate category) fitting into the commutative diagram

$$\begin{tikzcd} R \arrow[r, "p_2"] \arrow[d, "p_1"'] & X \arrow[d] \arrow[ddr] & \\ X \arrow[r] \arrow[drr] & X/R \arrow[dr, "\tau"] & \\ & & T \end{tikzcd}$$

The best result should be to know precisely when the universal quotient exist in some category (for example, in the category of algebraic schemes) and when it does not.

0.2. Particular Cases

1. The notion of algebraic space [3] which appears when morphisms $p_i, i = 1, 2$ assumed to be étale. In this case the object X/R belongs to the category of algebraic spaces.

2. Open covering of a scheme Y when $X = \bigsqcup U_\alpha$ is a disjoint union of open subschemes of Y, $R = \bigsqcup R_{\alpha\beta}$ and morphisms $R_{\alpha\beta} \to U_\alpha$ and $R_{\alpha\beta} \to U_\beta$ are local isomorphisms. In this case the object X/R is the scheme Y whose open cover was considered.

3. Let the scheme X is acted upon by an algebraic group G and R is a subscheme induced by G-equivalence [4]. In this case if $X/G = X/R$ exists its universality means that it is a categorical quotient of the scheme X by G.

0.3. Case of Several Components

Now let X be a disjoint union of schemes $X = X_1 \bigsqcup X_2$. Then

$$X \times X = X_1 \times X_1 \bigsqcup X_1 \times X_2 \bigsqcup X_2 \times X_1 \bigsqcup X_2 \times X_2$$

and the equivalence relation is broken into 4 disjoint components

$$R = R_{11} \bigsqcup R_{12} \bigsqcup R_{21} \bigsqcup R_{22}.$$

The problem reduces to the search of the universal completion of the diagram

$$\begin{tikzcd} R_{ij} \arrow[r, "p_j"] \arrow[d, "p_i"'] & X_j \\ X_i & \end{tikzcd}$$

This universal completion is called an *amalgam* (or an *amalgamated sum*) of schemes X_i and X_j with respect to R_{ij} and is denoted as $X_i \coprod_{R_{ij}} X_j$. If for each pair $i, j = 1, 2$ the amalgam $X_i \coprod_{R_{ij}} X_j$ exists then $X/R = X \coprod_R X = \coprod_{i,j}(X_i \coprod_{R_{ij}} X_j)$.

0.4. Schemes and algebras

We recall that affine Grothendieck' scheme of finite type over a field k is a prime spectrum $\operatorname{Spec} A$ of associative commutative algebra A of finite type with unity over k.

Recall that the k-algebra A is said to be of finite type if it admits a surjective homomorphism of polynomial algebra in finite set of variables $k[x_1, \dots, x_n] \twoheadrightarrow A$. In particular this means that the algebra A under consideration can have any finite Krull dimension (if A admits an epimorphism of polynomial algebra in n variables then the Krull dimension of A is not greater than n). In this case the algebra A as k-vector space can be infinitely-dimensional.

Generally, fibred product and amalgam (or product and corpoduct, or small limits) are dual categorical notions [5, 1.17,1.18] and are not obliged to exist in any category. The schemes under consideration are prime spectra of associative commutative algebras of finite type over k. Duality is done by functorial correspondence taking k-algebra to its prime spectrum. The fibred product of associative commutative k-algebras of finite type as taken by this functor to the amalgam of corresponding spectra. The functorial behavior of fibred product of algebras is analogous to one of fibred product of schemes.

Despite that existence of fibred product for associative commutative algebras is known, the method of explicit computation of it is not described in the literature. We fill this gap.

Let A be a commutative associative algebra of finite type with unity; then it can be represented as a quotient algebra of n-generated polynomial algebra $\phi : k[x_1, \dots, x_n] \twoheadrightarrow A$ where n depends on the structure of A. The representing homomorphism ϕ corresponds to the immersion of the scheme $\operatorname{Spec} A$ into k-affine space $\mathbb{A}^n = \operatorname{Spec} k[x_1, \dots, x_n]$. By means of this immersion we interpret the abstract k-scheme Z as a closed subscheme $Z \subset \mathbb{A}^n$ of affine space. We call the isomorphism $A \cong k[x_1, \dots, x_n]/\ker \phi$ (resp., the immersion $Z \subset \mathbb{A}^n$) the *affine representation* of the algebra A (resp., scheme Z). This is the reason why any associative commutative algebra of finite type with unity is called an *affine algebra*. Then the k-algebra is affine if and only if it has affine representation. Any affine algebra has many different affine representations.

We prove the following result.

Theorem 1. *The universal (fibred) product of affine algebras of finite type over a field can be compute explicitly by means of generators and relations.*

The method to compute fibred products of affine algebras constitutes the main subject of this article and described below.

First we fix the terminology and describe the algorithm which constructs algebra to be a universal product of algebras by means of their appropriate affine representations. Then we confirm that choice of different appropriate affine representations leads to isomorphic algebras. Finally we prove the universality and hence confirm that the obtained algebra is indeed a fibred product.

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1. Construction

1.1. Glossary

We work with polynomial rings of view $k[x_1, \dots, x_n]$, its ideals and quotients.

The *basis* of the ideal I is any system of generators of I as $k[x_1, \dots, x_n]$-module.

The basis is *minimal* if it does not span I whenever any of elements is excluded.

The *k-basis* of polynomial algebra (which is not obliged to be with unity) is its basis if the algebra is considered as k-vector space.

The *k-relation* between elements of the algebra A (which is not obliged to be with unity) is a nontrivial k-linear function (nontrivial k-linear combination) in these elements which equals 0 in A. As usually, linear combinations with finite number of nonzero coefficients are considered.

Fix the natural ordering of variables $x_1, \dots, x_n$; then we can use shorthand notation x^α for the monomial $x_1^{\alpha_1} \dots x_n^{\alpha_n}$. The symbol α denotes the row of degrees $\alpha_1, \dots, \alpha_n$.

1.2. Algorithm

Reduction to surjective morphisms

We are given three algebras A_0, A_1, A_2 with morphisms $A_1 \xrightarrow{f_1} A_0 \xleftarrow{f_2} A_2$. Show that we can assume that both f_i, $i = 1, 2$ are surjective. Since the morphisms f_i must include into commutative squares of the form

$$\begin{tikzcd} A_0 & A_2 \arrow[l, "f_2"'] \\ A_1 \arrow[u, "f_1"] & A_T \arrow[l, "\chi_1"] \arrow[u, "\chi_2"'] \end{tikzcd} \tag{1}$$

then for any such a commutative square the composite morphisms $f_1 \circ \chi_1$ and $f_2 \circ \chi_2$ have coincident images in A_0. By commutativity of the square $\operatorname{im} f_1 \circ \chi_1 = \operatorname{im} f_2 \circ \chi_2 \subset \operatorname{im} f_i$, $i = 1, 2$. This means that $\operatorname{im} f_i \circ \chi_i \subset \operatorname{im} f_1 \cap \operatorname{im} f_2$ and the morphism χ_i factors through the subalgebra $f_i^{-1}(\operatorname{im} f_1 \cap \operatorname{im} f_2) \subset A_i$ for $i = 1, 2$. Then all algebras of interest A_T complete the square

$$\begin{tikzcd} \operatorname{im} f_1 \cap \operatorname{im} f_2 & {f_2}^{-1}(\operatorname{im} f_1 \cap \operatorname{im} f_2) \arrow[l, two heads, "f_2'"'] \\ f_1^{-1}(\operatorname{im} f_1 \cap \operatorname{im} f_2) \arrow[u, two heads, "f_1'"] & A_T \arrow[l] \arrow[u] \end{tikzcd}$$

to commute.

Denote $A_1' \overset{f_1'}{\twoheadrightarrow} A_0' \overset{f_2'}{\twoheadleftarrow} A_2'$ where $A_0' = \operatorname{im} f_1 \cap \operatorname{im} f_2$, $A_i' = f_i^{-1}(A_0')$ and $f_i' = f_i|_{A_i'}$, $i = 1, 2$.

Let there is a fibred product $A'_1 \times_{A'_0} A'_2$ for the case $A'_1 \overset{f'_1}{\twoheadrightarrow} A'_0 \overset{f'_2}{\twoheadleftarrow} A'_2$. Confirm that $A'_1 \times_{A'_0} A'_2$ is also a universal product for the initial arbitrary data $A_1 \xrightarrow{f_1} A_0 \xleftarrow{f_2} A_2$. By the construction of algebras A'_i, $i = 0, 1, 2$, and by the definition of the algebra $A'_1 \times_{A'_0} A'_2$ it includes into the commutative diagram

$$\begin{tikzcd}
A_0 & & & A_2 \arrow[lll, "f_2"'] \\
& A'_0 \arrow[ul, hook] & A'_2 \arrow[l] \arrow[ur, hook] & \\
& A'_1 \arrow[u] \arrow[dl, hook] & A'_1 \times_{A'_0} A'_2 \arrow[l] \arrow[u] & \\
A_1 \arrow[uuu, "f_1"] & & &
\end{tikzcd}$$

Let A_T fit into the commutative diagram (1). Then by the universality of $A'_1 \times_{A'_0} A'_2$ as a product there is a unique homomorphism $\varphi : A_T \to A'_1 \times_{A'_0} A'_2$ which fits in the commutative diagram

$$\begin{tikzcd}
A_0 & & & A_2 \arrow[lll, "f_2"'] \\
& A'_0 \arrow[ul, hook] & A'_2 \arrow[l, "f'_2"'] \arrow[ur, hook] & \\
& A'_1 \arrow[u, "f'_1"] \arrow[dl, hook] & A'_1 \times_{A'_0} A'_2 \arrow[l] \arrow[u] & \\
A_1 \arrow[uuu, "f_1"] & & & A_T \arrow[lll] \arrow[uuu] \arrow[ull] \arrow[uul] \arrow[ul, dotted, "\varphi"']
\end{tikzcd}$$

This diagram provides universality of the algebra $A'_1 \times_{A'_0} A'_2$ as a product for initial data, i.e.

$$A'_1 \times_{A'_0} A'_2 = A_1 \times_{A_0} A_2.$$

Now we can replace everywhere our initial arbitrary data by the more special case when morphisms f_i, $i = 1, 2$, are surjective.

Easy case

Assume that $A_i \cong k[x_1, \dots, x_n]/I_i$, $i = 0, 1, 2$, so that $A_0 = A_1 \otimes_{k[x_1,\dots,x_n]} A_2$. This assumption says that the subscheme $\operatorname{Spec} A_0 \cong Z_0 \subset \mathbb{A}^n$ is an intersection of subschemes $Z_1 \subset \mathbb{A}^n$ and $Z_2 \subset \mathbb{A}^n$, $Z_i \cong \operatorname{Spec} A_i$, $i = 1, 2$.

Choose minimal bases in ideals I_i, $i = 1, 2$. Consider set of those monomials in $k[x_1, \dots, x_n]$ which are taken to 0 in both A_1 and A_2 by homomorphisms f_1 and f_2 respectively. It is clear that if x^α is such a monomial that for all $\beta \in \mathbb{Z}^n_{\geq 0}$ the monomial $x^{\alpha+\beta}$ is taken to 0 in both A_1, A_2. Then the set of monomials under consideration generates an ideal J.

Quotient algebra $k[x_1, \ldots, x_n]/J$ includes into the commutative square

$$\begin{tikzcd} A_0 & A_2 \arrow[l, two heads, "f_2"'] \\ A_1 \arrow[u, two heads, "f_1"] & k[x_1, \ldots, x_n]/J \arrow[l, two heads] \arrow[u, two heads] \end{tikzcd}$$

Consider k-basis elements in $k[x_1, \ldots, x_n]/J$. Since A_i are not obliged to be finite-dimensional over k, $k[x_1, \ldots, x_n]/J$ can also have infinite k-basis. We form the list L of k-relations among k-basis elements of $k[x_1, \ldots, x_n]/J$ as follows.

The k-relation $\sum a_\alpha x^\alpha$ between elements of $k[x_1, \ldots, x_n]/J$ (possibly monomial) is include into L if and only if it equals 0 in both of A_1, A_2. It is clear that the linear span $< L >$ is an ideal in $k[x_1, \ldots, x_n]/J$.

Now set $A = (k[x_1, \ldots, x_n]/J)/ < L >= k[x_1, \ldots, x_n]/(J+ < L >)$.

Remark 1. *By Hilbert's basis theorem, ideals J, $< L >$, and $J+ < L >$ admit finite bases.*

Remark 2. *If A_1 and A_2 finite-dimensional then* $\dim A_1 \times_{A_0} A_2 = \dim A_1 + \dim A_2 - \dim A_0$. *This follows immediately from the algorithm described. If k is algebraically closed then* $\operatorname{length} A_i = \dim A_i$, *$i = 0, 1, 2$, and then* $\operatorname{length} A_1 \times_{A_0} A_2 = \operatorname{length} A_1 + \operatorname{length} A_2 - \operatorname{length} A_0$.

Hard case

How to build up affine representations of algebras A_0, A_1, A_2 to validate the Easy case, i.e. such that $A_0 = A_1 \otimes_{k[x_1,\ldots,x_n]} A_2$?

Start with principal ideals in A_0: only those which are maximal under inclusion are necessary. Take a generator of each such an ideal. Choose any maximal k-linearly independent subset in the set of generators chosen. Since A_0 is an algebra of finite type, this subset is obliged to be finite; let it consist of s elements $g_1, \ldots, g_s$. Each element g_j corresponds to the variable x_j in $k[x_1, \ldots, x_s]$. The construction done fixes a homomorphism of k-algebras $h_0 : k[x_1, \ldots, x_s] \twoheadrightarrow A_0$. Let $K_0 := \ker h_0$. Since $k[x_1, \ldots, x_n]$ in Noetherian then K_0 is finitely generated ideal.

Let g'_j be one of preimages of the element $g_j \in A_0$ in A_1, g''_j be one of preimages of the same element in A_2. We complete the set $g'_1, \ldots, g'_s$ to form the set of k-linearly independent generators of the algebra A_1, by adding elements $g'_{s+1}, \ldots .g'_m \in A_1$. Similarly, the set $g''_1, \ldots, g''_s$ is completed to the set of k-linearly independent generators of the algebra A_2 by adding elements $g''_{m+1}, \ldots, g''_n$. We put the variables $x_{s+1}, \ldots, x_m$ in the correspondence to the elements $g'_{s+1}, \ldots, g'_m$ and the variables $x_{m+1}, \ldots, x_n$ to the elements $g''_{m+1}, \ldots, g''_n$.

This leads to homomorphisms $h_i : k[x_1, \ldots, x_n] \twoheadrightarrow A_i$, $i = 0, 1, 2$, $n \geq s$ defined by following correspondences

$$\begin{aligned} h_0(x_j) &= g_j, \quad j = 1, \ldots, s, \\ h_0(x_j) &= 0, \quad j = s+1, \ldots, n, \\ h_1(x_j) &= g'_j, \quad j = 1, \ldots, m, \\ h_1(x_j) &= 0, \quad j = m+1, \ldots, n, \\ h_2(x_j) &= g''_j, \quad j = 1, \ldots, s, m+1, \ldots, n, \\ h_2(x_j) &= 0, \quad j = s+1, \ldots, m. \end{aligned}$$

In particular, $h_0 = f_i \circ h_i$.

Affine representations of algebras A_0, A_1, A_2 and their homomorphisms f_1, f_2 lead to closed immersions of their spectra into affine space $\mathbb{A}^n$ according to the following commutative diagram

$$\begin{tikzcd} \mathbb{A}^n & \mathrm{Spec}\, A_2 \arrow[l, hook', "h_2^\sharp"'] \\ \mathrm{Spec}\, A_1 \arrow[u, hook, "h_1^\sharp"] & \mathrm{Spec}\, A_0 \arrow[l, hook', "f_1^\sharp"] \arrow[u, hook, "f_2^\sharp"'] \arrow[ul, hook', "h_0^\sharp"] \end{tikzcd}$$

By some modification of representing homomorphisms h_i, $i = 0, 1, 2$, we will easily achieve that in the appropriate affine space

$$h_0^\sharp(\mathrm{Spec}\, A_0) = h_1^\sharp(\mathrm{Spec}\, A_1) \cap h_2^\sharp(\mathrm{Spec}\, A_2).$$

For this purpose choose any system of generators $\kappa_1, \ldots, \kappa_r$ of the ideal K_0 (as $k[x_1, \ldots, x_n]$-module). Add to the set of variables $x_1, \ldots, x_n$ additional variables (whose number equals to the number r of generators if K_0 chosen) $y_1, \ldots, y_r$, and consider in A_1-algebra $A_1[y_1, \ldots, y_r]$ an ideal K'_0 generated by all relations of view $y_l - \kappa_l(g'_1, \ldots, g'_m)$, $l = 1, \ldots, r$. Then we have a commutative diagram

$$\begin{tikzcd} k[x_1, \ldots, x_n, y_1, \ldots, y_r] \arrow[r, two heads, "h'_2"] \arrow[d, two heads, "h'_1"'] \arrow[dr, two heads, "h'_0"] & A_2 \arrow[d, two heads, "f_2"] \\ A_1[y_1, \ldots y_r]/K'_0 \arrow[r, two heads] & A_0 \end{tikzcd}$$

There $h'_i(x_j) = h_i(x_j)$ for $j = 1, \ldots, n$ and for $i = 0, 1, 2$, but $h'_i(y_j) = 0$ for $i = 0, 2$ and $j = 1, \ldots, r$.

Lemma 1. *There is an isomorphism of k-algebras $A_1[y_1, \ldots y_r]/K'_0 \cong A_1$.*

Proof. Let $K_1 = \ker h_1$. Denote by J' the ideal in the ring $k[x_1, \ldots, x_n, y_1, \ldots, y_r]$ which is generated by relations $y_l - \kappa_l(x_1, \ldots, x_n)$, $l = 1, \ldots, r$. Obviously, $J' \subset \ker h'_1$. The isomorphism of $k[x_1, \ldots, x_n, y_1, \ldots, y_r]$-modules $A_1[y_1, \ldots y_r]/K'_0 \cong A_1$ follows from the exact diagram of $k[x_1, \ldots, x_n, y_1, \ldots y_r]$-modules

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & & \\
& J' \arrow[r, "\sim"] \arrow[d] & (y_1, \ldots, y_r) \arrow[d] & & \\
0 \arrow[r] & \ker h'_1 \arrow[r] \arrow[d] & k[x_1, \ldots, x_n, y_1, \ldots, y_r] \arrow[r] \arrow[d] & A_1[y_1, \ldots, y_r]/K'_0 \arrow[r] \arrow[d, "\wr"] & 0 \\
0 \arrow[r] & K_1 \arrow[r] \arrow[d] & k[x_1, \ldots, x_n] \arrow[r] \arrow[d] & A_1 \arrow[r] & 0 \\
& 0 & 0 & &
\end{tikzcd}$$

where the isomorphism $J' \cong (y_1, \ldots, y_r)$ is an isomorphism of free modules of equal ranks done by the correspondence $y_l - \kappa_l(x_1, \ldots, x_n) \mapsto y_l$, $l = 1, \ldots, r$. The k-algebra isomorphism $A_1[y_1, \ldots y_r]/K'_0 \cong A_1$ follows from the fact that the quotient algebra $A_1[y_1, \ldots y_r]/K'_0$ admits the same system of generators $g'_1, \ldots, g'_m$ as an algebra A_1, with same relations (equal to generators of the ideal K_1). $\square$

Proposition 1. *The homomorphisms h'_i lead to the expression*

$$A_0 = A_1 \otimes_{k[x_1, \ldots, x_n, y_1, \ldots, y_r]} A_2.$$

Proof. To prove the proposition one needs to confirm that A_0 is universal as a coproduct. Let Q be a k-algebra supplied with two homomorphisms $A_1 \xrightarrow{\varphi_1} Q \xleftarrow{\varphi_2} A_2$ such that the following diagram commutes:

$$\begin{tikzcd}
k[x_1, \ldots, x_n, y_1, \ldots, y_r] \arrow[r, two heads, "h'_2"] \arrow[d, two heads, "h'_1"'] & A_2 \arrow[d, two heads, "f_2"'] \arrow[dr, "\varphi_2"] & \\
A_1 \arrow[r, two heads, "f_1"] \arrow[rrd, "\varphi_1"'] & A_0 & \\
& & Q
\end{tikzcd}$$

Commutativity of the ambient contour guarantees that $\varphi_1(A_1) = \varphi_2(A_2)$ and hence we can replace Q by $\varphi_1(A_1) = \varphi_2(A_2)$. However we preserve the notation Q but assume that homomorphisms φ_i are surjective. By the construction of homomorphisms h'_i, the algebra A_0 is generated by the images of first s variables $x_1, \ldots, x_s$.

Then by the commutativity of the ambient contour $\varphi_1 h'_1(x_j) = \varphi_2 h'_2(x_j)$ for $j = 1, \ldots, n$. In particular, by the construction of homomorphisms h_i (and of homomorphisms h'_i built up by means of them), $\varphi_1 h'_1(x_j) = \varphi_2 h'_2(x_j) = 0$ for $j = s+1, \ldots, n$. For $j = 1, \ldots s$ we have $0 = \varphi_1 \circ h'_1(y_l - \kappa_l(x_j)) = \varphi_2 \circ h'_2(y_l - \kappa_l(x_j)) = -\varphi_2 h'_2 \kappa_l(x_j)$ for $l = 1, \ldots r$ since $h'_2(y_l) = 0$. This implies that homomorphisms φ_i, $i = 1, 2$, factor through the algebra A_0.

The homomorphism $\varphi_0 : A_0 \to Q$ is uniquely defined on generators $h'_0(x_j)$ of A_0 by the correspondence $h_0(x_j) \mapsto \varphi_i \circ h'_i(x_j)$, $i = 1, 2$, $j = 1, \ldots, s$. $\square$

Remark 3. *Proposition 1 means that affine representations constructed are such that the intersection of images of schemes* $\operatorname{Spec} A_1$ *and* $\operatorname{Spec} A_2$ *under closed immersions into* $\operatorname{Spec} k[x_1, \dots, x_n, y_1, \dots, y_r] = \mathbb{A}^{n+r}$ *is the image of* $\operatorname{Spec} A_0$.

Remark 4. *In further considerations we use the notation of the view*

$$A_1 \xleftarrow{h_1} k[x_1, \dots, x_n] \xrightarrow{h_2} A_2$$

assuming that homomorphisms h_i. $i = 1, 2$, are surjective and that they admit application of the Easy case, i.e. Proposition 1 holds for them.

2. Well-Definedness and Universality

2.1. Different affine representations chosen

Proposition 2. *Different affine representations of algebras A_0, A_1, A_2 lead to isomorphic product algebras A.*

Proof. Let we have two pairs of different affine representations defined by pairs of homomorphisms

$$A_1 \overset{h_1}{\twoheadleftarrow} k[x_1, \dots, x_n] \overset{h_2}{\twoheadrightarrow} A_2 \quad \text{and} \quad A_1 \overset{h'_1}{\twoheadleftarrow} k[x'_1, \dots, x'_m] \overset{h'_2}{\twoheadrightarrow} A_2$$

such that $A_0 = A_1 \otimes_{k[x'_1, \dots, x'_m]} A_2 = A_1 \otimes_{k[x_1, \dots, x_n]} A_2$. They correspond to two pairs of closed immersions of prime spectra

$$\operatorname{Spec} A_1 \overset{h_1^\sharp}{\hookrightarrow} \mathbb{A}^n \overset{h_2^\sharp}{\hookleftarrow} \operatorname{Spec} A_2 \quad \text{and} \quad \operatorname{Spec} A_1 \overset{h_1'^\sharp}{\hookrightarrow} \mathbb{A}^m \overset{h_2'^\sharp}{\hookleftarrow} \operatorname{Spec} A_2$$

such that $h_1^\sharp(\operatorname{Spec} A_1) \cap h_2^\sharp(\operatorname{Spec} A_2) = h_0^\sharp(\operatorname{Spec} A_0)$ and $h_1'^\sharp(\operatorname{Spec} A_1) \cap h_2'^\sharp(\operatorname{Spec} A_2) = h_0'^\sharp(\operatorname{Spec} A_0)$. This leads to closed immersions $\overline{h_i^\sharp} : \operatorname{Spec} A_i \hookrightarrow \mathbb{A}^n \times \mathbb{A}^m = \mathbb{A}^{n+m}$ as composites with the diagonal immersion diag :

$$\overline{h_i^\sharp} : \operatorname{Spec} A_i \overset{\text{diag}}{\hookrightarrow} \operatorname{Spec} A_i \times \operatorname{Spec} A_i \xrightarrow{(h_i^\sharp, h_i'^\sharp)} \mathbb{A}^n \times \mathbb{A}^m.$$

Now confirm that in this case also $\overline{h_1^\sharp}(\operatorname{Spec} A_1) \cap \overline{h_2^\sharp}(\operatorname{Spec} A_2) = \overline{h_0^\sharp}(\operatorname{Spec} A_0)$.

Homomorphisms $\overline{h}_i$ are defined as composites

$$\begin{array}{ccccc} k[x_1, \dots, x_n] \otimes_k k[x'_1, \dots, x'_m] & \xrightarrow{(h_i, h'_i)} & A_i \otimes_k A_i & \overset{\delta_i}{\to} & A_i, \\ g_1 \otimes g_2 & \mapsto & h_i(g_1) \otimes h'_i(g_2) & \mapsto & h_i(g_1) \cdot h'_i(g_2). \end{array}$$

Let Q be an algebra together with two homomorphisms $\varphi_i : A_i \to Q$ such that $\varphi_1 \circ \overline{h}_1 = \varphi_2 \circ \overline{h}_2$. To confirm that $A_0 = A_1 \otimes_{k[x_1, \dots, x_n] \otimes_k k[x'_1, \dots, x'_m]} A_2$ it is necessary to construct a unique homomorphism $\varphi_0 : A_0 \to A_T$ such that $\varphi_i = \varphi_0 \circ f_i$. Consider commutative diagrams

$$\begin{tikzcd}
A_i \otimes_k A_i \arrow[r, "(f_i, f_i)"] \arrow[d, "\delta_i"'] & A_0 \otimes_k A_0 \arrow[d, "\delta_0"] \arrow[ddr, "\varphi'_0"] & \\
A_i \arrow[r, "f_i"] \arrow[drr, "\varphi_i"'] & A_0 \arrow[dr, dotted, "\varphi_0"] & \\
 & & Q
\end{tikzcd}$$

for $i = 1, 2$. Let for any reducible tensor $x \otimes x' \in A_i \otimes_k A_i$

$$\begin{array}{ll} (f_i, f_i)(x \otimes x') = \overline{x} \otimes \overline{x}' \in A_0 \otimes_k A_0; & \delta_0(\overline{x} \otimes \overline{x}') = \overline{x} \cdot \overline{x}' \in A_0; \\ f_i(x \cdot x') = \overline{x} \otimes \overline{x}' \in A_0; & \delta_i(x \otimes x') = x \cdot x' \in A_i. \end{array}$$

Since by universality of the product

$$A_0 \otimes_k A_0 = (A_1 \otimes_k A_1) \otimes_{k[x_1,\dots,x_n] \otimes_k k[x'_1,\dots,x'_m]} (A_2 \otimes_k A_2)$$

the homomorphism φ'_0 is uniquely defined using homomorphisms

$$(\varphi_i, \varphi_i) : A_i \otimes_k A_i \to Q : x \otimes x' \mapsto \varphi_i(x) \cdot \varphi_i(x')$$

as

$$\varphi'_0(\overline{x} \otimes \overline{x}') = \varphi_i(x) \cdot \varphi_i(x') \in Q$$

then $\varphi_0 : A_0 \to Q$ is also uniquely defined as $\varphi_0(\overline{y}) = \varphi_i(y)$ for $\overline{y} = f_i(y)$, $y \in A_i$.

Now it rests to form product algebras for three different affine representations

$$\begin{tikzcd} k[x_1,\dots,x_n] \arrow[d, "h_i"'] & k[x_1,\dots,x_n] \otimes_k k[x'_1,\dots,x'_m] \arrow[l, two heads, "\lambda"'] \arrow[r, two heads, "\rho"] \arrow[d, "\overline{h}_i"'] & k[x'_1,\dots,x'_m] \arrow[d, "h'_i"] \\ A_i \arrow[r, equal] & A_i \arrow[r, equal] & A_i \end{tikzcd}$$

Horizontal homomorphisms are defined by correspondences $\lambda : x'_l \mapsto 0$, $l = 1, \dots, m$ and $\rho : x_j \mapsto 0$, $j = 1, \dots, n$ respectively.

Denoting by $\overline{A}$ the product algebra for $\overline{h}_i : k[x_1,\dots,x_n] \otimes_k k[x'_1,\dots,x'_m] \twoheadrightarrow A_i$, $i = 0, 1, 2$, we come to the commutative diagram

$$\begin{tikzcd} k[x_1,\dots,x_n] \arrow[d, "h"'] & k[x_1,\dots,x_n] \otimes_k k[x'_1,\dots,x'_m] \arrow[l, two heads, "\lambda"'] \arrow[r, two heads, "\rho"] \arrow[d, "\overline{h}"'] & k[x'_1,\dots,x'_m] \arrow[d, "h'"] \\ A & \overline{A} \arrow[l, two heads, "p_A"'] \arrow[r, two heads, "p'_A"] & A' \end{tikzcd} \tag{2}$$

associated to closed immersions

$$\begin{array}{ccccc} \mathbb{A}^n & \hookrightarrow & \mathbb{A}^n \times \mathbb{A}^m & \hookleftarrow & \mathbb{A}^m, \\ (x_1,\dots,x_n) & \longmapsto & (x_1,\dots,x_n,0,\dots,0), & & \\ & & (0,\dots,0,x'_1,\dots,x'_m) & \longleftarrow\!\shortmid & (x'_1,\dots,x'_m). \end{array}$$

Vertical homomorphisms in (2) define affine representations for product algebras built up according to the Easy case. Lower horizontal morphisms are surjective by commutativity of the diagram (2).

To prove isomorphicity of lower homomorphisms in (2) consider sections of homomorphisms $k[x_1,\dots,x_n] \overset{\lambda}{\twoheadleftarrow} k[x_1,\dots,x_n] \otimes_k k[x'_1,\dots,x'_m] \overset{\rho}{\twoheadrightarrow} k[x_1,\dots,x_m]$, i.e. homomorphisms $k[x_1,\dots,x_n] \overset{s_1}{\hookrightarrow} k[x_1,\dots,x_n] \otimes_k k[x'_1,\dots,x'_m] \overset{s_2}{\hookleftarrow} k[x_1,\dots,x_m]$ defined by following rules $s_1 : f \mapsto f \otimes 1$, $s_2 : g \mapsto 1 \otimes g$. A nonzero element from A has nonzero preimage in $k[x_1,\dots,x_n]$ which has nonzero image in $k[x_1,\dots,x_n] \otimes_k k[x'_1,\dots,x'_m]$. This image is taken to nonzero element of at least one of algebras A_1, A_2 and hence has nonzero image in $\overline{A}$. This leads to a homomorphism of k-algebras $s_A : A \to \overline{A}$. It is a section of the homomorphism p_A by its construction.

Now remark that s_A is surjective. Indeed, one can choose a system of k-generators (which is not obliged to be a basis) in $\overline{A}$ which are images of reducible tensors $f \otimes g$ from $k[x_1, \dots, x_n] \otimes_k k[x'_1, \dots, x'_m]$, $f \in k[x_1, \dots, x_n]$, $g \in k[x'_1, \dots, x'_m]$. This expression is defined up to the action of $k^* = k \setminus 0$. A polynomial f has nonzero image at least in one of algebras A_1, A_2. Hence it is taken to nonzero element in A. Since $p_A \circ s_A = \mathrm{id}_A$ then both s_A and p_A are isomorphisms.

Isomorphicity of p'_A is proven analogously. $\square$

2.2. Universality

Proposition 3. *The construction of product algebra A is indeed universal, i.e. A is true fibred product.*

Proof. Let A_T be an affine k-algebra together with two homomorphisms $\chi_i : A_T \to A_i$, $i = 1, 2$ such that $\chi_1 \circ f_1 = \chi_2 \circ f_2$. We construct a homomorphism φ to complete the diagram

$$\begin{tikzcd}
& A_T \arrow[dl, "\chi_1"'] \arrow[d, dotted, "\varphi"] \arrow[dr, "\chi_2"] & \\
A_1 & A \arrow[l, "\overline{f}_1"'] \arrow[r, "\overline{f}_2"] & A_2
\end{tikzcd}$$

Choose appropriate affine representations of algebras A_T, A, A_i, $i = 1, 2$. Perform the manipulations as described in the Easy case. Namely, let J be an ideal in $k[x_1, \dots, x_n]$ generated by all monomials taken to 0 in A_T, A and A_i, $i = 1, 2$.

Quotient algebra $k[x_1, \dots, x_n]/J$ includes in the commutative diagram

$$\begin{tikzcd}
& A_1 \arrow[dl, "f_1"'] & & \\
A_0 & A_T \arrow[u, "\chi_1"'] \arrow[d, "\chi_2"] & A \arrow[ul, "\overline{f}_1"'] \arrow[dl, "\overline{f}_2"] & k[x_1, \dots, x_n]/J \arrow[l, two heads] \arrow[ull] \arrow[dll] \\
& A_2 \arrow[ul, "f_2"] & &
\end{tikzcd}$$

Choose an arbitrary element $\alpha \in A_T$; it is taken to $\chi_1(\alpha) \in A_1$ and to $\chi_2(\alpha) \in A_2$ so that $f_1\chi_1(\alpha) = f_2\chi_2(\alpha)$. Any γ of preimages of α in $k[x_1, \dots, x_n]/J$ is also taken to $f_1\chi_1(\alpha) = f_2\chi_2(\alpha) \in A_0$ and to some element $\overline{\alpha} \in A$. We claim that $\overline{\alpha}$ does not depend on the choice of γ.

Choose another $\gamma' \in k[x_1, \dots, x_n]/J$ taken to α, then $\gamma - \gamma'$ maps to zero in A_T and hence it is mapped to zero in both A_i, $i = 1, 2$. Then by the construction of A (cf. Easy case) $\gamma - \gamma'$ is taken to zero in A. This shows that there is a homomorphism $\varphi : A_T \to A$ such that $\chi_i = \varphi \circ \overline{f}_i$, $i = 1, 2$, as required.

It rests to confirm ourselves that the homomorphism $\varphi : A_T \to A$ does not depend on the choice of affine representations of algebras A_T, A_i, $i = 0, 1, 2$. For this sake assume that there are two different collections of affine representations include into commutative diagrams

$$
\begin{tikzcd}
 & k[x_1,\dots,x_n] \arrow[ddl, "h_1"'] \arrow[d, "h_T"] \arrow[ddr, "h_2"] & \\
 & A_T \arrow[dl] \arrow[dr] & \\
A_1 \arrow[dr, "f_1"'] & A \arrow[l] \arrow[r] & A_2 \arrow[dl, "f_2"] \\
 & A_0 &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & k[x'_1,\dots,x'_m] \arrow[ddl, "h'_1"'] \arrow[d, "h'_T"] \arrow[ddr, "h'_2"] & \\
 & A_T \arrow[dl] \arrow[dr] & \\
A_1 \arrow[dr, "f_1"'] & A \arrow[l] \arrow[r] & A_2 \arrow[dl, "f_2"] \\
 & A_0 &
\end{tikzcd}
\tag{3}
$$

Then consider the algebra $k[x_1,\dots,x_n]\otimes_k k[x'_1,\dots,x'_m]$ and its homomorphisms on quotient algebras $k[x_1,\dots,x_n] \xleftarrow{\lambda} k[x_1,\dots,x_n]\otimes_k k[x'_1,\dots,x'_m] \xrightarrow{\rho} k[x'_1,\dots,x'_m]$ which are defined by their kernels $\ker\lambda = (x'_1,\dots,x'_m)$, $\ker\rho = (x_1,\dots,x_n)$ and correspond to immersions of subspaces $\mathbb{A}^n \overset{\lambda^\sharp}{\hookrightarrow} \mathbb{A}^n\times\mathbb{A}^m \overset{\rho^\sharp}{\hookleftarrow} \mathbb{A}^m$ as linear subvarieties. Representations (3) define induced affine representations for algebras A_T, A_i, $i=0,1,2$, according to the following rule: $\overline{h}_T(x_j\otimes x'_l) = h_T(x_j)\cdot h'_T(x'_l)$, $\overline{h}_i(x_j\otimes x'_l) = h_i(x_j)\cdot h'_i(x'_l)$, $i=0,1,2$, $j=1,\dots,n$, $l=1,\dots,m$. This rule corresponds to the composites of immersions of prime spectra

$$
\begin{aligned}
\overline{h}_T^\sharp &: \operatorname{Spec} A_T \overset{\text{diag}}{\hookrightarrow} \operatorname{Spec} A_T\times\operatorname{Spec} A_T \overset{(h_T^\sharp,{h'_T}^\sharp)}{\hookrightarrow} \mathbb{A}^n\times\mathbb{A}^m,\\
\overline{h}_i^\sharp &: \operatorname{Spec} A_i \overset{\text{diag}}{\hookrightarrow} \operatorname{Spec} A_i\times\operatorname{Spec} A_i \overset{(h_i^\sharp,{h'_i}^\sharp)}{\hookrightarrow} \mathbb{A}^n\times\mathbb{A}^m.
\end{aligned}
$$

Let J be the ideal generated by all monomials in $k[x_1,\dots,x_n]$ taken to 0 in A_T and in A_i, $i=0,1,2$, J' be the similar ideal in $k[x'_1,\dots,x'_m]$, and $\overline{J}$ the similar ideal in $k[x_1,\dots,x_n]\otimes_k k[x'_1,\dots,x'_m]$. Quotient algebras by these ideals include into the following commutative diagram

$$
\begin{tikzcd}
k[x_1,\dots,x_n] \arrow[d] & k[x_1,\dots,x_n]\otimes_k k[x'_1,\dots,x'_m] \arrow[l, "\lambda"'] \arrow[r, "\rho"] \arrow[d] & k[x'_1,\dots,x'_m] \arrow[d] \\
k[x_1,\dots,x_n]/J & k[x_1,\dots,x_n]\otimes_k k[x'_1,\dots,x'_m]/\overline{J} \arrow[l, two heads] \arrow[r, two heads] & k[x'_1,\dots,x'_m]/J'
\end{tikzcd}
$$

Now we perform the construction of homomorphisms φ,φ' and $\overline{\varphi}$ using quotient algebras $k[x_1,\dots,x_n]/J, k[x'_1,\dots,x'_m]/J'$ and $k[x_1,\dots,x_n]\otimes_k k[x'_1,\dots,x'_m]/\overline{J}$ respectively, as described in the beginning of this subsection. This yields in two commutative diagrams

$$
\begin{tikzcd}
k[x_1,\dots,x_n]/J \arrow[dd] \arrow[dr] & & k[x_1,\dots,x_n]\otimes_k k[x'_1,\dots,x'_m]/\overline{J} \arrow[ll, two heads] \arrow[dd] \arrow[dr] & \\
 & A_T \arrow[rr, equal] \arrow[dl, "\varphi"'] & & A_T \arrow[dl, "\overline{\varphi}"'] \\
A \arrow[rr, equal] & & A &
\end{tikzcd}
$$

$$\begin{tikzcd}
k[x_1,\dots,x_n]\otimes_k k[x'_1,\dots,x'_m]/\overline{J} \arrow[rr, two heads] \arrow[dd] \arrow[dr, two heads] & & k[x'_1,\dots,x'_m]/J' \arrow[dd] \arrow[dr, two heads] & \\
& A_T \arrow[rr, equal] \arrow[dl, "\overline{\varphi}"'] & & A_T \arrow[dl, "\varphi'"] \\
A \arrow[rr, equal] & & A &
\end{tikzcd}$$

what implies that $\varphi = \varphi' = \overline{\varphi}$. $\square$

Example 1. *For example consider $A_1 = A_2 = k[x,y]/(x^2-y)$, $A_0 = k[x]/(x^2)$, with homomorphisms $f_i : x \mapsto x, y \mapsto 0$, $i = 1,2$. The "occasional" affine representation used in the definition of these algebras cannot be involved to construct their product since $A_1 \otimes_{k[x,y]} A_2 = k[x,y]/(x^2-y) \ne A_0$. Then it is necessary to form a new affine representation which fits for constructing a product, using Hard case.*

The algebra $A_0 = k[x]/(x^2)$ contains unique maximal principal ideal. Its generating element x corresponds to first variable x_1 and gives rise to the affine representation $k[x_1] \twoheadrightarrow A_0$, $x_1 \mapsto x$. Further, manipulations according to the Hard case lead to the polynomial algebra $k[x_1,x_2,x_3,y]$ and representations for algebras

$$A_1 = k[x_1,x_2,x_3,y]/(x_1^2-x_2, x_3, x_1^2-y), \quad A_2 = k[x_1,x_2,x_3,y]/(x_1^2-x_3, x_2, y).$$

These algebras represent geometrically two parabolas with common tangent line in two 2-dimensional planes in 4-dimensional affine space. Planes meet along this tangent line. In this case

$$\begin{aligned} A_1 \otimes_{k[x_1,x_2,x_3,y]} A_2 &= k[x_1,x_2,x_3,y]/(x_1^2-x_2, x_3, x_1^2-y, x_1^2-x_3, x_2, y) \\ &= k[x_1,x_2,x_3,y]/(x_1^2, x_2, x_3, y) = A_0. \end{aligned}$$

This means that our two parabolas intersect along the subscheme defined by the algebra A_0. This validates application of Easy case to affine representations we've constructed. The list J of monomials vanishing in both algebras A_i, $i = 1,2$, is empty. The ideal $< L >$ has a form $< L > = (x_1^2 - x_2 - x_3, x_2 - y, x_2x_3)$, and $A_1 \times_{A_0} A_2 = k[x_1,x_2,x_3,y]/ < L > = k[x_1,x_2,x_3,y]/(x_1^2-x_2-x_3, x_2-y, x_2x_3) = k[x_1,x_2,x_3]/(x_1^2-x_2-x_3, x_2x_3)$. From the geometrical point of view this is union of two parabolas having a common tangent line at the origin. Parabolas lie in different 2-planes which meet along parabolas' common tangent line.

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Аннотация. В работе дан и обоснован метод прямого вычисления универсального (расслоенного) произведения в категории коммутативных ассоциативных алгебр конечного типа с единицей над полем. Поле коэффициентов не предполагается алгебраически замкнутым и может иметь любую характеристику. Формирование расслоенного произведения коммутативных ассоциативных алгебр составляет алгебраическую сторону процедуры склеивания алгебраических схем по некоторому отношению эквивалентности в алгебраической геометрии. Если исходные алгебры являются конечномерными векторными пространствами, то размерность их расслоенного произведения подчиняется формуле, аналогичной формуле размерности суммы подпространств. Геометрически конечномерный случай поставляет строгую версию объединения двух наборов точек, имеющих общую часть. Метод использует задание алгебр образующими и определяющими соотношениями на входе и выдает аналогичное представление произведения на выходе. Он пригоден для компьютерной реализации. Произведение алгебр определено корректно: выбор иных представлений тех же алгебр приводит к изоморфной алгебре-произведению. Также показано, что алгебра-произведеное обладает свойством универсальности, т.е. является настоящим расслоенным произведением. Входные данные – это тройка алгебр и пара гомоморфизмов $A_1 \stackrel{f_1}{\to} A_0 \stackrel{f_2}{\leftarrow} A_2$. Алгебры и гомоморфизмы могут быть заданы произвольным образом. Показано, что для вычисления расслоенного произведения достаточно ограничиться случаем, когда гомоморфизмы $f_i, i = 1, 2$ сюръективны, и описан способ редукции к сюръективному случаю. Также рассмотрено правило выбора образующих и соотношений для исходных алгебр.

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