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Edge States and Chiral Solitons in Topological Hall and Chern–Simons Fields

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Abstract. The multi-component extension problem of the $(2+1)D$-gauge topological Jackiw–Pi model describing the nonlinear quantum dynamics of charged particles in multi-layer Hall systems is considered. By applying the dimensional reduction $(2+1)D \to (1+1)D$ to Lagrangians with the Chern–Simons topologic fields , multi-component nonlinear Schrodinger equations for particles are constructed with allowance for their interaction. With Hirota's method, an exact two-soliton solution is obtained, which is of interest in quantum information transmission systems due to the stability of their propagation. An asymptotic analysis $t \to \pm\infty$ of soliton-soliton interactions shows that there is no backscattering processes. We identify these solutions with the edge (topological protected) states – chiral solitons – in the multi-layer quantum Hall systems. By applying the Hirota bilinear operator algebra and a current theorem, it is shown that, in contrast to the usual vector solitons, the dynamics of new solutions (chiral vector solitons) has exclusively unidirectional motion. The article is published in the author's wording.

Keywords: chiral solitons, Chern–Simons fields, topological fields, nonlinear Schrödinger equation, fractional quantum Hall effect.

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1. Introduction

The (2+1)D matter particles interacting with gauge topological Chern–Simons fields support solitons solutions [1, 2, 3, 4, 5, 6]. Solitons are a central paradigm in many branches of present-day physics and mathematics. One of most interesting combination of the quantum and nonlinear properties of solitons arises in the (2+1)D-nonlinear Schrödinger equation gauged by a Chern–Simons fields (Jackiw–Pi model) [4, 5]. The model [4], proposed by Jackiw and Pi is described by the (2+1)D –action

$$S_{(2+1)\mathrm{D}} = \int \left(\frac{1}{\overline{\kappa}} \varepsilon^{\mu\nu\rho} F_{\mu\nu} A_\rho + i\hbar \psi^* D_t \psi - \frac{\hbar^2}{2m} |D_i \psi|^2 - V(\rho) \right) d^3x, \tag{1}$$

where $\psi \in \mathbb{C}$ is the Schrödinger quantum field, giving rise to charged bosonic particles after second quantization. A_μ is the Chern–Simons gauge field, $V(\rho)$ is the self-interaction potential of charged particles, $\rho = \psi^*\psi$ is the particle density and $D_\mu = \partial_\mu - iA_\mu$ $(\mu = 0, 1, 2)$ is the covariant derivative. The metric tensor is $g^{\mu\nu} = (1, -1, -1)$ and $\varepsilon^{\mu\nu\rho}$ is the Levi-Civita anti-symmetric symbol.

In context of the Landay–Ginzburg mean field theory the Langrange density of model (1) can be considered as the model to describe the *edge states* (chiral solitons) of the Integer Quantum Hall Effects in monolayer systems [6]. In this case the scalar field ψ is the order parameter and the constant $\overline{\kappa}$ is interpreted as the Hall conductivity. In the 2D-system of charged particles the Hall quantum current is given by $J_x = \sigma_{xy} E_y$, where the transverse conductivity σ_{xy} can be computed as:

$$\sigma_{xy} = \frac{e^2}{2\pi h} \int dk_x dk_y \Omega(k_x, k_y) = \frac{\gamma_B}{R_{K-90}}. \tag{2}$$

Here h is the Planck constant, $\Omega = \mathrm{rot} A$, $(\Omega_{ij} = \partial_i A_j - \partial_j A_i$ is the Berry curvature), γ_B is Berry holonomy (phase), k_j is a wave-number and $R_{K-90} = (h/e^2) = 25812,807\Omega_{90}$ is the Klitzing practical standard for Electrical Resistance, used in resistance calibrations worldwide. The quantum Hall resistance $R_H = \alpha c/2\pi$ also provides an extremely independent determination of the fine structure constant $\alpha-$ the quantity of fundamental importance in QED(Quantum Electro-Dynamics, c is the maximum speed at which all known form of information in the universe can travel).

As well know [2, 6] in the multi-layer systems to take place the Fractional Quantum Hall Effect due to inter-layer correlations of interacting anyons – the planar particles with unconventional statistics. So it is interesting the extension of the theory (1) to the multi-component case of matter field: $\psi \to \psi_j$, $j = 1, 2, \ldots N$.

In this Letter we present a multi-component generalization of the Jackiw–Pi models (1). As the (2+1)D-second-order field equations associated to (1) are not integrable, it is natural to consider a dimensional reduction $L_{(2+1)\mathrm{D}} \to L_{(1+1)\mathrm{D}}$.

2. Lagrangian and the dimensional reduction $L_{(2+1)\mathrm{D}} \to L_{(1+1)\mathrm{D}}$

We start by suppressing dependence on the one space-like coordinate and redefining the gauge field as $A_2 = (mc/\hbar^2)B$, $A_1 = A_1$ and $A_0 = A_0 + (mc/\hbar^2)B$. By adding suitable kinetic term for B-field we have reduced (1) to the following total (1+1)D-Lagrangian

$$L_{tot(1+1)} = L_{(1+1)} + L_B + L_{BF}, \tag{3}$$

where

$$L_B = \frac{g}{2\hbar}\dot{B}B', \;\; L_{BF} = (2\kappa)^{-1}B\varepsilon^{\alpha\beta}F_{\alpha\beta}. \tag{4}$$

Here dot/prime indicate differentiation with respect to time/space, g is the coupling constant of B-fields, $\kappa = (\hbar^2/mc)\overline{\kappa}$ is dimensionless and we have neglected the term $\partial_x(B^3/3\hbar\kappa)$ since it is a total spatial derivative. $V = V(\overline{\psi}\psi)$ is a general polynomial in the density $|\psi_j|^2$ and describes nonlinear self-actions and inter-actions between N components of matter fields ψ_j, $j = 1, 2, \ldots, N$.

After elimination of the gauge fields (A_μ, B) by using its dynamic eq. and phase redefinition of ψ_j we obtain from (3) a final local-invariant Lagrangian

$$L_{(1+1)\mathrm{D}} = \int dx \sum_{j=1}^{N} \left(i\hbar\overline{\psi}_j\partial_t\psi_j - H_j - V\right), \tag{5}$$

$$H_j = \frac{\hbar^2}{2m}\overline{\Pi}_j\Pi_j,\; \Pi_j = \left(\partial_x \pm i\kappa^2\sum_k^N \rho_k\right)\psi_j,\; \rho_k = \overline{\psi_k}\psi_k.$$

So the Euler–Lagrange equation reads

$$\left(i\partial_\tau + D_\zeta^2 - g\kappa^2\sum_k^N J_k\right)\psi_j = \frac{\partial V}{\partial\overline{\psi}_j}. \tag{6}$$

Here $\tau = \hbar^{-1}t$ and $\zeta = x(\hbar^2/2m)^{-1/2}$ is the normalized time and space variables, $D_\zeta = \partial_\zeta + ig\kappa^2\rho$ is the gauge covariant derivative, $\kappa = (\hbar^2/mc)\overline{\kappa}$ is the Hall dimensionless constant which are connected to the Berry phase γ_B and called as a Chern (number) topological invariant. At the same times the factor κ is defined by Hall conductivity: $\sigma_{xy} = (e^2/h)\nu = \kappa/2\pi$, where ν is the filled factor of Landau energy levels. In the Integer QHE (IQHE)$-\nu \in \mathbb{Z}$. In the Fractional QHE (FQHE)$-\nu \in \mathbb{Q}$.

The particle density $\rho = \sum_k^N \rho_k$ and the total current $J = \sum_k^N J_k$, where $J_k = \frac{1}{2i}(\overline{\psi}_k D_\zeta\psi_k - \psi_k\overline{D_\zeta\psi_k})$, satisfy the *continuity equation*: $\rho_{,\tau} + J_{,\zeta} = 0$. Jackiw and Pi found that the dynamics of model (1) is not Lorentz-invariant [1, 2, 3, 4]. Also it is not Galileo-invariant. Another interesting fact is the nontrivial gauge analogy between (1+1)D multi-component model (6) and standard nonlinear Schrödinger equations with $SU(m,n)$ group symmetry [7, 8]. A simple analysis allows us to describe an exact solution for system Eqs. (6).

3. Nonlinear Schrödinger equation and two-component chiral solitons

Let us first assume that $V = 0$. Now if we define a nonlocal transformation [5, 9]

$$\psi_j = \exp\left[-ig\kappa^2\int_0^\zeta d\zeta'\rho(\zeta')\right]\tilde{\psi}_j \tag{7}$$

and use also the continuity Eq. we obtain modified multi-component nonlinear Schrödinger equation (the sign tilde omit below)

$$iL_0\psi_j = 2g\kappa^2 \left(\sum_k^N J_k\right)\psi_j, \quad (L_0 = \partial_\tau - i\partial_{\zeta\zeta}) \tag{8}$$

with a current-nonlinearity

$$J_k = \frac{1}{2i}\left(\overline{\psi}_k \psi_{k,\zeta} - \psi_k \overline{\psi}_{k,\zeta}\right). \tag{9}$$

This is to be contrasted with the familiar multi-component (vector) $U(m,n)$ nonlinear Schrödinger equation [7, 10]:

$$-iL_0\psi_j = 2\left(\sum_k^N \lambda_k |\psi_k|^2\right)\psi_j, \tag{10}$$

where to take place the usual charge density nonlinearity $|\psi_k|^2 = \rho_k$. At the same times we note that our Eq. (8) is nothing but the multi-component (vector) generalization of Aglietti et al eq. [5].

Influenced by the known solutions [7] to Eq. (10), we can be obtained an exact solution to Eq. (8) by the Hirota bilinear method [10, 11]:

$$\psi_j = G_j/H, \ H \in \mathbb{R}, \ j = 1, 2, \dots N. \tag{11}$$

In the simple case $N = 2$ the solution to (8) which has one soliton for each component is given by

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = H^{-1} \begin{pmatrix} \gamma_1 e^{\sigma_1}\left(1 + a_{22}b_{12}\exp(\sigma_2 + \overline{\sigma}_2)\right) \\ \gamma_2 e^{\sigma_2}\left(1 + a_{11}b_{21}\exp(\sigma_1 + \overline{\sigma}_1)\right) \end{pmatrix}, \tag{12}$$

where the Hirota function has the form

$$H = 1 + \sum_k^2 a_{kk} e^{\sigma_k + \overline{\sigma}_k} + a_{11}a_{22}b_{12}b_{21}e^{\sigma_1+\overline{\sigma}_1+\sigma_2+\overline{\sigma}_2} \tag{13}$$

with

$$\sigma_\mu = k_\mu \zeta + ik_\mu^2 \tau, \ a_{nn} = \frac{\lambda_n |\gamma_n|^2}{(k_n + \overline{k}_n)^2}, \ b_{nm} = \frac{k_n - k_m}{k_n + \overline{k}_m}, \tag{14}$$

and $\overline{a} = a^*$($*-$complex conjugation). The parameters k_j related to the amplitudes, width and velocity of the j-th soliton [7, 10].

4. Asymptotic $\tau \to \pm\infty$ analysis of the two-soliton solutions

Evidently that the two-soliton solution (12) describe a process of soliton - soliton scattering for the inter-component interaction. The asymptotic ($\tau \to \pm\infty$) analysis show that

$$\psi_j(\zeta, \tau \to \pm\infty) = \sum_n^N C_j^{n\pm} \mathrm{sech}\left[\frac{U_j}{2}(\zeta - V_n\tau + \zeta_{nj}^{\pm})\right] e^{i\Theta_{nj}}, \tag{15}$$

where $U_n = 2\mathrm{Re}\,k_n$ and $V_n = 2\mathrm{Im}\,k_n$ are the inverse width and velocity of n-th soliton, respectively,

$$\zeta_{11}^{-} = \ln\frac{a_{11}}{k_{11}},\ \zeta_{22}^{-} = \ln\frac{a_{22}\left|b_{21}\right|^2}{k_{22}},\ \zeta_{22}^{+} = \ln\frac{a_{22}}{k_{22}},\ \zeta_{11}^{+} = \ln\frac{a_{11}\left|b_{12}\right|^2}{k_{11}}$$

is the soliton phase before $(-)$ and after $(+)$ the interaction. The soliton amplitudes C_j^{n-} and C_j^{n+} are related to one another as

$$C_j^{n+} = T_j^n C_j^{n-},\ T_1^1 = \frac{b_{12}}{|b_{12}|},\ T_2^2 = \frac{b_{21}}{|b_{21}|}. \tag{16}$$

It is seen that $IT_j^n I^2 = I$ and the interaction-induced phase shifts of solitons $\Delta_j = \zeta_{jn}^{+} - \zeta_{jn}^{-}$ obey the Sudzuki-Zakharov-Shabat condition: $\zeta_{11}\Delta_1 - \zeta_{22}\Delta_2$ (conservation law for the soliton center of mass) [7, 8]. Now for establishing the integrability property of the our system (8) we propose a following theorem

Theorem: *If the current* (9) *satisfy* to condition $J_k = V_k\rho_k/2$, then the form (12) is an exact two-soliton solution of Eq. (8).

Proof of theorem: The proof this theorem is ordinary. Using the Hirota bilinear operator D, defined as $D(U{\cdot}V) = (DU)V - U(DV)$, by direct computation the current (9) on the basis of the form (12) we find

$$J_k = \frac{i}{2}\frac{D_\zeta G_k \circ G_k}{H^2} = \frac{1}{2}V_k\left|\psi_k\right|^2 \tag{17}$$

iff the condition $\mathrm{Im}(k_n - k_m) = 0$ is satisfied.

5. Conclusion

As (17), the our Eq. (8) becomes identical to Eq. (10) with

$$\lambda_k = -\frac{1}{2}g\kappa^2 V_k. \tag{18}$$

The interesting pattern is follow from condition of (18): if $\lambda_k > 0$ with the gauge coupling constant fixed $(g > 0)$, then the soliton in all components moves in one direction $(V_k < 0)$ – it is *the chiral solitons*. This is contrasted to the usual NLS Eq. (10), whose solitons can move in both directions. Finally, we obtain one unexpected result – the $(1+1)D$-multi-component chiral solitons is *NO-scattering* composite particles on the Line, as $V_1 = V_2$.

We plan to obtain edge modes solution in model (1) for the non-hermitian case, by operator dressing method [7, 12].

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Агаларов А. М., Гаджимурадов Т. А., Потапов А. А., Рассадин А. Э., "Краевые состояния и киральные солитоны в топологических полях Черна–Саймонса–Холла", *Моделирование и анализ информационных систем*, **25**:1 (2018), 133–139.

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Аннотация. Рассматривается проблема многокомпонентного расширения $(2+1)D$-калибровочной топологической модели Jackiw–Pi, описывающей нелинейную квантовую динамику заряженных частиц в многослойных системах Холла. Применяя размерную редукцию $(2+1)D \to (1+1)D$ к лагранжианам с топологическими полями Черна–Саймонса, мы построили многокомпонентные нелинейные уравнения Шредингера для частиц с учетом их взаимодействия. Используя метод Хироты, получили точное двухсолитонное решение, представляющее интерес для квантовых систем передачи информации в силу устойчивости их распространения. Асимптотический $t \to \pm\infty$ анализ солитон-солитонных взаимодействий показывает, что процессов обратного рассеяния нет. Мы отождествляем эти решения с краевыми (топологически защищенными) состояниями – киральными солитонами – в многослойных квантовых системах Холла. Применяя билинейную операторную алгебру Хироты и теорему тока, мы показали, что в отличие от обычных векторных солитонов динамика новых решений (киральных векторных солитонов) имеет исключительно однонаправленное движение. Статья публикуется в авторской редакции.

Ключевые слова: векторные киральные солитоны, поле Черна–Саймонса, топологическое поле, нелинейное уравнение Шредингера, дробный квантовый эффект Холла.

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