Continuous Flattening of a Regular Tetrahedron with Explicit Mappings

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We proved in [10] that each Platonic polyhedron \( P \) can be folded into a flat multilayered face of \( P \) by a continuous folding process of polyhedra. In this paper, we give explicit formulas of continuous functions for such a continuous flattening process in \( \mathbb{R}^3 \) for a regular tetrahedron.

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1. Introduction

We use the terminology *polyhedron* for a closed polyhedral surface which is permitted to touch itself but not self-intersect (and so a doubly covered polygon is a polyhedron). A *flat folding* of a polyhedron is a folding by creases into a multilayered planar shape ([7], [8]).

A. Cauchy [4] in 1813 proved that any convex polyhedron is rigid: precisely, if two convex polyhedra \( P, P' \) are combinatorially equivalent and their corresponding faces are congruent, then \( P \) and \( P' \) are congruent. By removing the condition of convexity, R. Connelly [5] in 1978 gave an example of a (non-convex) flexible polyhedron: precisely, there is a continuous family of polyhedra \( \{P_t : 0 \leq t \leq 1\} \) such that for every \( t \neq 0 \), the corresponding faces of \( P_0 \) and \( P_t \) are congruent while polyhedra \( P_0 \) and \( P_t \) are not congruent. (See also [6].) After then I. Sabitov [15] in 1998 proved that the volume of any polyhedron is invariant under flexing: precisely, if there is a continuous family of polyhedra \( \{P_t : 0 \leq t \leq 1\} \) such that, for every \( t \), the corresponding faces of \( P_0 \) and \( P_t \) are congruent, then the volumes \( P_0 \) and \( P_t \) are equal for all \( 0 \leq t \leq 1 \). (See also [14].)

A. Milka [12] in 1994 showed that any polyhedron admits a continuous (isometric) deformation by using moving edges, and that all Platonic polyhedra can be changed in

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their exterior shapes. He called such a deformation a linear bending. I. Sabitov [16] in 2000 explained on a class of deformations of polyhedra without the face-rigidity condition, and he introduced an example of a regular tetrahedron whose vertex is pushed in continuously with moving creases (which he called "swimming" edges). Also T. Banchoff [2] discussed on a continuous isometric deformation of a square polyhedral torus. D. Bleeker [3] showed that any convex polyhedron in \( \mathbb{R}^3 \) admits a continuous isometric deformation such that the volume increases. (See also [13].)

In our previous works, we discussed on flattening of convex polyhedra by a continuous (isometric) deformation which we call a continuous folding process (see Definition 1). For such a deformation, its faces must be changed by moving creases as its volume decreases. We proved in 2010 that each Platonic polyhedron is fattened by a continuous flat folding process of polyhedra onto its original face ([10]). Furthermore we showed with C. Vîlcu in 2011 that any convex polyhedron admits infinitely many continuous folding processes, by using cut loci and Alexandrov’s gluing theorem ([1], [11]).

In this paper, we give explicit formulas of continuous functions for a continuous flat folding process in the case of a regular tetrahedron. We leave such calculation for other Platonic polyhedra in future work.

2. Explicit formulas of continuous flat folding mappings

We proved the following lemma which played a key role for proofs in [10].

**Lemma.** For any \( l (0 \leq l \leq |AC|) \) and any \( m (0 \leq l \leq |BD|) \) of a rhombus \( ABCD \) with the center \( M \), there are points \( Q, R \) on the line segment \( AC \) which satisfy the following:

(i) \( |QM| = |RM| \),
(ii) by folding \( R \) with creases \( \{AM, BM, DM, RB, RD, MR, RC\} \), \( \triangle BMR \) and \( \triangle DMR \) are folded onto \( \triangle BQM \) and \( \triangle DQM \) respectively, and
(iii) \( \text{dist}\{A', C'\} = l \) and \( \text{dist}\{B', D'\} = m \), where \( X' \) in the folded rhombus is the corresponding point to a point \( X \) in the rhombus \( ABCD \) and \( |YZ| \) means the length of a line segment \( YZ \). (see Fig. 1).

**Definition 1.** Let \( P \) be a polyhedron in the Euclidean space \( \mathbb{R}^3 \). We say that a family of polyhedra \( \{P_t : 0 \leq t \leq 1\} \) is a continuous folding process from \( P = P_0 \) to \( P_1 \) if it satisfies the following conditions:

(1) for each \( 0 \leq t \leq 1 \), there exists a polyhedron \( P'_t \) obtained from \( P \) by subdividing some faces of \( P \) (i.e., some faces of \( P'_t \) may be included in the same face of \( P \), but \( P'_t \) is congruent to \( P \)) such that \( P_t \) is combinatorially equivalent to \( P'_t \) and the corresponding faces of \( P'_t \) and \( P_t \) are congruent,

(2) the mapping \( [0, 1] \ni t \mapsto P_t \in \{P_t : 0 \leq t \leq 1\} \) is continuous.

**Theorem.** The regular tetrahedron in \( \mathbb{R}^3 \) with vertices \( O = (0, 0, 0) \), \( A = (2/\sqrt{3}, 0, -2\sqrt{2}/\sqrt{3}) \), \( B = (\sqrt{3}, 1, 0) \), and \( C = (\sqrt{3}, -1, 0) \) is flattened explicitly by a continuous folding process of polyhedra \( \{P_t : 0 \leq t \leq 1\} \) which satisfies the following:

(i) the line segment \( OM \), where \( M \) is the midpoint of the edge \( BC \), is fixed on the \( x \)-axis (see Fig. 2(1)(2)),
(ii) two faces \( \triangle OAB \) and \( \triangle OAC \) have no crease,
Figure 1. How to fold a rhombus: (1) a rhombus (2) an example of a folded rhombus.

Figure 2. (1) A tetrahedron (2) the tetrahedron with coordinates.
(iii) there are points $Q_t (0 \leq t \leq 1)$ on $OM$ and $R_t$ on $AM$ such that for each $t$ the face $\triangle ABC$ is divided into four triangles $\triangle ABR_t$, $\triangle ACR_t$, $\triangle BCR_t$, and $\triangle CR_t$, and that $\triangle B_t(R_t)'M$ and $\triangle C_t(R_t)'M$ are attached to $\triangle B_tQ_tM$ and $\triangle C_tQ_tM$ respectively, where we denote by $A_t$, $B_t$, $C_t$ and $(R_t)'$ the points on $P_t$ corresponding to points $A$, $B$, $C$ and $R_t$ respectively (see Fig. 3), and

(iv) explicit coordinates of $A_t$, $B_t$, $C_t$ and $Q_t$ are

$$
A_t = \left( \frac{6 + 2s\sqrt{6} + 3s^2}{\sqrt{3}(3 + s^2)}, 0, \frac{2(s - \sqrt{6} + 3s^2)}{3 + s^2} \right),
$$

$$
B_t = (\sqrt{3}, \sqrt{1 - s^2}, s),
$$

$$
C_t = (\sqrt{3}, -\sqrt{1 - s^2}, s),
$$

$$
Q_t = \left( \frac{\sqrt{3}(3 + s^2)}{3 + s\sqrt{6} + 3s^2}, 0, 0 \right)
$$

where $s = \sin \frac{\pi}{2}t$ $(0 \leq t \leq 1)$ (see Figure 2 and Figure 3), and

(v) $P_1$ is a flat folded state of $P$ (see Fig. 4).

Figure 3. $P_t (0 \leq t \leq 1)$ where $s = \sin \frac{\pi}{2}t$.

Proof. Let $P$ be a regular tetrahedron with the edge length two, whose vertices are $O, A, B, C$ in the $xyz$-space $\mathbb{R}^3$ with coordinates $O = (0, 0, 0)$, $A = (2/\sqrt{3}, 0, -2\sqrt{2}/\sqrt{3})$, $B = (\sqrt{3}, 1, 0)$, and $C = (\sqrt{3}, -1, 0)$. Denote by $M = (\sqrt{3}, 0, 0)$ the midpoint of the edge $BC$.

Get a crease of the line segment $OM$ on the face $\triangle OBC$, fold $\triangle OBC$ continuously into halves until $B$ and $C$ meets on the $xz$-plane. According to such folding, we define a
Figure 4. The flat folded state $P_t$ of the regular tetrahedron $P$.

Continuous folding process of $P_t$ ($1 \leq t \leq t$) for $P$ which satisfies the following conditions (i) – (v) in Theorem. Fix the line segment $OM$ for all $0 \leq t \leq 1$ and denote $s(t) = \sin \frac{\pi}{2} t$ ($0 \leq t \leq 1$). Rotate triangle $OBM$ and triangle $OCM$ about $OM$ as follows: for each $t$ ($0 \leq t \leq 1$)

$$B_t = (\sqrt{3}, \sqrt{1-s^2}, s),$$

$$C_t = (\sqrt{3}, -\sqrt{1-s^2}, s),$$

where $s = s(t) = \sin \frac{\pi}{2}$ and by $X_t \in P_t$ (for $X = A, B, C$) and $(R_t)'$ the corresponding point to points $X$ and $R_t$ on $P$.

Since we get no crease on $\triangle OAB$ and $\triangle OAC$, the distances between $O$, $A_t$ and $A_t$, $B_t$ are $\text{dist}(O, A_t) = 2$ and $\text{dist}(A_t, B_t) = 2$. Then the coordinates of $A = (x_t, 0, z_t)$ satisfy

$$\begin{align*}
(x_t)^2 + (z_t)^2 &= 4 \quad (1) \\
(\sqrt{3}-x_t)^2 + (1-s^2) + (s-z_t)^2 &= 4 \quad (2)
\end{align*}$$

By subtracting (2) from (1) in each side of the equalities, it follows

$$(x_t)^2 + (z_t)^2 - \{(\sqrt{3}-x_t)^2 + (1-s^2) + (s-z_t)^2\} = 0.$$  

Hence,

$$x_t = \frac{2-s \cdot z_t}{\sqrt{3}}. \quad (3)$$

Substituting the equation (3) to the equation (1), by $z_t \leq 0$

$$z_t = \frac{2(s-\sqrt{6+3s^2})}{3+s^2}, \quad (4)$$

and hence

$$A_t = \left(\frac{6+2s\sqrt{6+3s^2}}{\sqrt{3}(3+s^2)}, 0, \frac{2(s-\sqrt{6+3s^2})}{3+s^2}\right).$$
By Lemma 1 note that \( Q_t \) is the intersection point of the orthogonal bisector of \( OA_t \) with the line segment \( OM \). Let \( Q_t = (q_t, 0, 0) \). Since the midpoint \( N_t \) of \( OA_t \) is \( N_t = (x_t, 0, z_t) \) and the inner product of the vector \( OA \) and the vector \( N_tQ_t \) is zero, we have

\[
\left(\frac{x_t}{2} - q_t\right) \cdot x_t + \frac{(z_t)^2}{2} = 0.
\]

By (1) it holds

\[
q_t = \frac{2}{x_t}.
\]

Therefore we get

\[
Q_t = \left(\frac{\sqrt{3}(3 + s^2)}{3 + s\sqrt{6} + 3s^2}, 0, 0\right).
\]

\[\square\]

3. Another continuous flattening

In Theorem 1, we pushed the face \( \triangle ABC \) inside to flatten the regular tetrahedron \( OABC \) (see Fig. 3 and Fig. 4). If we push the face \( \triangle ABC \) outside, can we still flatten the tetrahedron continuously? We show there is a continuous folding process for such a flattening. Denote by \( G \) and \( H \) the centers of gravity for \( \triangle ABC \) and \( \triangle OAB \) (see Fig. 5(1)). Then the quadrilateral \( HBGC \) is a rhombus which is folded in a halfway with a crease \( BC \) (see Fig. 5(2)). By applying Lemma 1, there are moving creases \( BR_t \) and \( CR_t \) \((0 \leq t \leq 1)\) with points \( R_t \) on the line segment \( MG \) such that \( \text{dist}\{B_t, C_t\} \) decreases to zero and \( \text{dist}\{H, (R_t)’\} \) increases to \( 2/\sqrt{3} \) simultaneously for given distances (see Fig. 5(3) and Fig. 3(4)) where we use the same notations for \( A_t, B_t, C_t \) and \( (R_t)’ \) as the ones used in the proof of Theorem 1.

Let fold \( \triangle OAB \) and \( \triangle OAC \) similar way to the one defined in the proof of Theorem 1. Then a continuous folding process of \( P_t \) \((0 \leq t \leq 1)\) is obtained as follows:

(i) the coordinates of \( G_t \in P_t \) (which is the corresponding point to \( G \)) is uniquely determined by the equation (see Fig. 6)

\[
|A_tG_t| = |B_tG_t| = |C_tG_t| = 2/\sqrt{3},
\]

(ii) by using similar argument to the one used in the proof of Theorem 1, we can calculate the coordinates of \( (R_t)’ \in P_t \).

Finally the tetrahedron is flattened in a shape showed in Fig. 7.

4. Further research

It is almost obvious that the area of creases for the continuous flattening process discussed in the section 3 is smaller than the one showed in Theorem 1. We asked the following problem in [10]
Question. What is the minimum area of creases which are used for a continuous flattening of each Platonic polyhedron?

For the case of flattening showed in Theorem 1, the area is $1/\sqrt{3}$, and for the one showed in the section 3, Ko-ichi Hirata [9] got an approximate value $(\sqrt{3} - \sqrt{2})/2$ by using Mathematica.

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References

Figure 6. Another continuous folding process of $P_t$ ($0 \leq t \leq 1$) where $s = \sin \frac{\pi}{2} t$.

Figure 7. The flat folded state $P_1$ by another flattening.


Непрерывное уплощение правильного тетраэдра точными отображениями

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Ключевые слова: непрерывное складывание, правильные тетраэдры, многогранники, складывание бумаги

В статье [10] нами доказано, что любой правильный многогранник $P$ допускает непрерывное (изометричное) складывание (или разглаживание) на плоскость. В настоящей статье мы приводим явные формулы непрерывных функций такого процесса складывания для правильного тетраэдра в $\mathbb{R}^3$. Статья публикуется в авторской редакции.

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