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On Homology Groups of a Subspace of Triangulations of the Two-Simplex with not More than 6 Subdivisional Boundary Vertices

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We study homology groups of the space $\tilde{W}_1(\nabla^N)$ of triangulations of the two-dimensional simplex with vertices $D_0D_1D_2$ endowed with a boundary subdivision with not more than 6 vertices in case when this boundary subdivision is extended to the interior of the simplex without adding new interior vertices. As a result, we obtain a theorem about the homology groups H_n in cases $n = 0, \dots, 5$.

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Introduction

Let us consider the two-dimensional simplex σ^2 with vertices $D_0D_1D_2$ endowed with a boundary subdivision that contains three old vertices of the simplex. Assume that this subdivision contains at most N new vertices and, moreover, that the edge D_0D_1 contains at most p new vertices, D_1D_2 contains at most q new vertices, D_2D_0 contains at most s new vertices, where p, q, s are non-negative integers, $p + q + s = n$, $0 \leq n \leq N$. Such a subdivision will be said to be of type (p, q, s) . Here we are considering the triangulations of the simplex of the special type. The boundary subdivision is extended to the interior of the simplex without adding new interior vertices and the obtained triangulations don't contain the interior subsimplex with the vertices lying on all three edges of σ^2 .

Our goal is to find the homology groups of the space of these triangulations.

1. Construction of the space of triangulations

Let us identify the set of the boundary subdivision of the simplex σ^2 with a polyhedron $\nabla_{p,q,s}^n$. As a result we have the natural surjective mapping

$$\pi : Tr(p, q, s) \rightarrow \nabla_{p,q,s}^n,$$

assigning to any triangulations the corresponding boundary subdivision. It is not bijective since there are many ways of extending a given boundary subdivision to the interior of the simplex. Thus we may regard $Tr(p, q, s)$ as a (ramified) finite sheeted covering $\tilde{\nabla}_{p,q,s}^n$ over each polyhedron $\nabla_{p,q,s}^n$. It was proved in [1] that the number T_k of different triangulations lying over an open k -cell $\overset{\circ}{\nabla}_{p,q,s}^k$ ($p + q + s = k$) of the polyhedron $\nabla_{p,q,s}^n$ in the case when we don't consider the triangulations with the interior subsimplex with the vertices on all three edges of σ^2 is given by the formula

$$T_k = C_{k-1}^{p-1} + C_{k-1}^{q-1} + C_{k-1}^{s-1}.$$

(Note that $T_0 = T_1 = 1$.)

The space of such triangulations of the 2-dimensional simplex σ^2 is a cell complex $\tilde{W}_1(\nabla^N)$.

2. Construction of the incidence matrixies

In order to find the homology groups let us use the chain complex $C = \oplus_n C_n$ corresponding to our CW-space and construct the incidence matrixies ${}^n E$, defying the boundary mapping $C_{n+1} \rightarrow C_n$. Left index of the incidence matrix contains all n -dimensional cells (sheets of the covering) of the polyhedron $\tilde{\nabla}_{p,q,s}^n$ ($p + q + s = n$, $p, q, s \geq 0$), upper index contains all $(n + 1)$ -dimensional cells (sheets) of the polyhedron $\tilde{\nabla}_{p,q,s}^{n+1}$ ($p + q + s = n + 1$, $p, q, s \geq 0$), each cell has a fixed orientation. In the intesection of the line i and column j we put the incidence coefficient of the $(n + 1)$ -dimensional j -cell and n -dimensional i -cell. The number of the cell is defined in such a way that the incidence matrix has the most simple block structure. It has non-zero blocks on the left-right diagonal of the matrix and some non-zero blocks near this diagonal. To get this block structure we had to define the special order of the n -dimensional cells (sheets) in the basis of the group C_n .

The first cells in the basis of C_n are $\nabla_{n,0,0}^n, \nabla_{0,n,0}^n, \nabla_{0,0,n}^n$, consisting the block ${}^n \mathcal{A}$. The next block of cells ${}^{n-1} \mathcal{A}$ contains sheets of the coverings $\tilde{\nabla}_{n-p,p,0}^n$ and $\tilde{\nabla}_{p,n-p,0}^n$ ($1 \leq p \leq k$, $2k = n, 2k + 1 = n$) in order

$$\tilde{\nabla}_{n-1,1,0}^n, \tilde{\nabla}_{1,n-1,0}^n, \tilde{\nabla}_{n-2,2,0}^n, \tilde{\nabla}_{2,n-2,0}^n, \dots, \tilde{\nabla}_{k,k,0}^n; \tag{1}$$

then the sheets of the coverings $\tilde{\nabla}_{n-p,0,p}^n$ and $\tilde{\nabla}_{p,0,n-p}^n$ ($1 \leq p \leq k$) and at last – the sheets of the coverings $\tilde{\nabla}_{0,n-p,p}^n$ and $\tilde{\nabla}_{0,p,n-p}^n$ ($1 \leq p \leq k$.) (The order of these sheets is similar to (1)). Each covering $\tilde{\nabla}_{p,q,0}^n(\nabla_{p,0,q}^n, \tilde{\nabla}_{0,p,q}^n)$ $p + q = n$, $p, q \geq 0$, contains C_n^p n -dimensional cells (sheets) of the CW-space $\tilde{W}_1(\nabla^N)$. These cells are numbered in the following way.

In [1] it has been proved that each extension of a given boundary subdivision to the interior of the simplex σ^2 is defined by the weights of the subdivisional boundary vertices. By the weight of the boundary vertice we call the number of the interior divisional segments connecting this vertice with other divisional vertices of the boundary. In case when we have subdivisional vertices only on the two edges of the simplex the sum of the weights on one of the two edges of the simplex is equal to n . Consequently the first

n -cells in the basis of C_n are defined by the following sequences of weights:

$$\begin{aligned}
 &(n - p + 1, 1, \dots, 1; v_1^{(1)}, \dots, v_q^{(1)}; 0) \\
 &(1, n - p + 1, \dots, 1; v_1^{(2)}, \dots, v_q^{(2)}; 0) \\
 &\dots\dots\dots \\
 &(1, 1, \dots, n - p + 1; v_1^{(p)}, \dots, v_q^{(p)}; 0).
 \end{aligned} \tag{2}$$

The next n -cells are defined by the following sequences of weights:

$$\begin{aligned}
 &(n - p, 2, \dots, 1; \tilde{v}_1^{(1)}, \dots, \tilde{v}_q^{(1)}; 0) \\
 &(n - p, 1, 2, \dots, 1; \tilde{v}_1^{(2)}, \dots, \tilde{v}_q^{(2)}; 0) \\
 &\dots\dots\dots \\
 &(n - p, 1, \dots, 2; \tilde{v}_1^{(p-1)}, \dots, \tilde{v}_q^{(p-1)}; 0) \\
 &(2, n - p, 1, \dots, 1; \tilde{v}_1^{(p)}, \dots, \tilde{v}_q^{(p)}; 0) \\
 &(1, n - p, 2, \dots, 1; \tilde{v}_1^{(p+1)}, \dots, \tilde{v}_q^{(p+1)}; 0) \\
 &\dots\dots\dots \\
 &(1, 1, \dots, 2, n - p; \tilde{v}_1^{(p^2-p)}, \dots, \tilde{v}_q^{(p^2-p)}; 0).
 \end{aligned} \tag{3}$$

The next n -cells are defined by the sequences of weights where one of them is equal to $n - p - 1$, one is equal to 3 and all others on this edge have a weight 1. And so on we are numbering the cells in the same way, i.e. deducting u_1 by 1, fixing the set u_2, u_3, \dots, u_p so that $u_1 + \dots + u_p = n$, $u_1 \geq u_2 \geq \dots \geq u_p$ and considering all various rearrangments of the set. Then the same process we apply to the set of the weights v_1, \dots, v_q .

The next set of blocks of n -cells in the basis of C_n are blocks of type (p, q, s) , $p, q, s > 0$:

$$\tilde{\nabla}_{p,q,s}^n, \tilde{\nabla}_{p,s,q}^n, \tilde{\nabla}_{q,p,s}^n, \tilde{\nabla}_{s,p,q}^n, \tilde{\nabla}_{q,s,p}^n, \tilde{\nabla}_{s,q,p}^n, \quad p + q + s = n, \quad p \geq q \geq s.$$

These blocks of cells follow in the order of lessening of q . In each covering $\tilde{\nabla}_{p,q,s}^n$ ($p+q+s = n, p, q, s > 0$) we are numbering n -cells corresponding to the triangulations of the simplex σ^2 with maximum sum of the weights of subdivisioanal vertices of one of the edges. This part contains $C_{n-1}^{p-1} + C_{n-1}^{q-1} + C_{n-1}^{s-1}$ n -cells (sheets) of $\tilde{W}_1(\nabla^N)$. The cells are numbered in the same way as in the block ${}^n_{n-1}\mathcal{A}$. The only difference here: we have subdivisioanal vertices on all the edges, so we have to consider the third set of the weights t_1, \dots, t_s . This third set changes in the same way as the first two sets.

The basis of the group C_{n+1} is constructed in the same way.

The main difficulty here is the dimensions of the incidence matrix. Even if n is small the martix ${}^n E$ is quite big. ${}^0 E$ is a zero matrix with one line and 3 columns. ${}^1 E$ is a 3×9 matrix:

$$\begin{pmatrix}
 -1 & 0 & 0 & 0 & -1 & 0 & 1 & & \\
 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & -1 \\
 & & -1 & 0 & 0 & 1 & 0 & -1 &
 \end{pmatrix}.$$

$A_{3j} - 10 \times 15$, $A_{4j} - 10 \times 16 (j = 1, \dots, 6)$. B and C are block matrixes

$$\begin{pmatrix} B_1 & & & & & \\ B_2 & & & & & \\ & B_3 & & & & \\ & B_4 & & & & \\ & & B_5 & & & \\ & & & B_6 & & \end{pmatrix}, \quad \begin{pmatrix} C_1 & & & & & \\ & C_3 & & & & \\ & C_2 & & & & \\ & & & & C_5 & \\ & & & C_4 & & \\ & & & & & C_6 \end{pmatrix},$$

where $B_i (i = 1, \dots, 6) - 5 \times 12$, $C_i (i = 1, \dots, 6) - 10 \times 20$ matrixies.

J and F have the same block structure:

$$\begin{pmatrix} F_1 & F_2 & & & & \\ & & F_3 & F_4 & & \\ & & & & F_5 & F_6 \end{pmatrix},$$

where $F_i (i = 1, \dots, 6)$ is a 8×16 matrix, $J_i (i = 1, \dots, 6) - 1 \times 6$ matrix. G has the following block structure:

$$\begin{pmatrix} G_1 & & G_3 & & & \\ & G_2 & & G_5 & & \\ & & G_4 & & G_6 & \end{pmatrix},$$

where each G_i is a 9×16 matrix. H is a column of three 9×15 matrixies.

3. Homology groups of the space of triangulations

We use the following elementary transformations of the columns of the incidence matrix ${}^{n-1}E$:

- 1) exchanging of the places of the columns i and j ;
- 2) multiplying the column i by -1 ;
- 3) adding the column j to the column i .

The coordinated transformations of the incidence matrix nE are:

- 1) exchanging of the lines i and j ;
- 2) multiplying the line i by -1 ;
- 3) subtracting the line i from the line j .

As a result of the coordinated transformations of the incidence matrixies ${}^0E, {}^1E, \dots, {}^5E$ of the CW-space $\tilde{W}_1(\nabla^N)$ we obtain the ranks of these matrixies:

$$\text{rang } {}^1E = 3, \quad \text{rang } {}^2E = 6, \quad \text{rang } {}^3E = 18, \quad \text{rang } {}^4E = 42, \quad \text{rang } {}^5E = 102$$

and the zero torsion coefficients.

Now we can calculate the Betti numbers with the help of the formula

$$p_i = \alpha_i - \text{rang } {}^{i-1}E - \text{rang } {}^iE,$$

where α_i is the number of the i -dimensional cells of the CW-complex:

$$p_1 = 0, \quad p_2 = 0, \quad p_3 = 0, \quad p_4 = 0, \quad p_5 = 0.$$

Theorem 1. *Let $\tilde{W}_1(\nabla^N)$ be a CW-space of triangulations of the 2-simplex σ^2 with at most N subdivisional vertices on its edges. Assume that the boundary subdivision is extended to the interior of σ^2 without adding new interior vertices and triangulations don't contain interior subsimplex with the vertices on the three edges of the simplex. Then the homology groups H_n of this space are trivial in cases $n = 1, 2, 3, 4, 5$ and in case $n = 0$ the homology group is a free cyclic group.*

If we consider the space of triangulations of the simplex σ^2 with the interior subsimplex τ^2 with vertices on all edges of σ^2 then the dimensions of the incidence matrixes increase, because the number of different triangulations lying over an open k -cell $\overset{\circ}{\nabla}_{p,q,s}^k$ ($p + q + s = k$) of the polyhedron $\nabla_{p,q,s}^n$ is given by the formula

$$T_k = C_{k-1}^{p-1} + C_{k-1}^{q-1} + C_{k-1}^{s-1} + \sum_{i=1}^p \sum_{j=1}^q C_{p-i+j-1}^{p-i} C_{q+s+i-j-1}^{s-1}.$$

In nE we must include n -cells corresponding to these triangulations. They follow after the cells corresponding to the triangulations without subsimplex τ^2 . The order of these n -cells depends on the position of the vertices of subsimplex τ^2 . At first we include the cells corresponding to the triangulations of σ^2 where two first vertices of τ^2 are the first subdivisional vertices of the edges D_0D_1 and D_1D_2 accordingly and the third vertex is going in succession through all subdivisional vertices of the edge D_2D_0 . The numbering of n -cells corresponding to the triangulations with the fixed third vertex of τ^2 is the same as it was described earlier. Then we move the second vertex of τ^2 to the second subdivisional vertex of the edge D_1D_2 and once more allow the third vertex to move through all subdivisional vertices of the edge D_2D_0 and so on. After using all subdivisional vertices on D_1D_2 as the second vertex of τ^2 we move the first vertex of τ^2 from the first subdivisional vertex of D_0D_1 to the second and begin the same process again.

In this case the dimensions of the incidence matrixes are: ${}^0E - 1 \times 3$, ${}^1E - 3 \times 9$, ${}^2E - 9 \times 16$, ${}^3E - 16 \times 66$, ${}^4E - 66 \times 168$, ${}^5E - 168 \times 416$.

As a result of the coordinated transformations of the incidence matrixes ${}^0E, {}^1E, \dots, {}^5E$ of the CW-space $W_1(\nabla^N)$ we obtain Betti numbers

$$p_1 = 0, \quad p_2 = 0, \quad p_3 = 1, \quad p_4 = 0, \quad p_5 = 0$$

and the zero torsion coefficients.

Theorem 2. *Let $W_1(\nabla^N)$ be a CW-space of triangulations of the 2-simplex σ^2 with at most N subdivisional vertices on its edges. Assume that the boundary subdivision is extended to the interior of σ^2 without adding new interior vertices. Then the homology groups H_n of this space are trivial in cases $n = 1, 2, 4, 5$ and in cases $n = 0, 3$ the homology group is a free cyclic group.[2]*

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О группах гомологий одного подпространства триангуляций двумерного симплекса с не более чем 6 точками разбиения границы

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Рассматривается группа гомологий пространства $\tilde{W}_1(\nabla^N)$ триангуляций двумерного симплекса $D_0D_1D_2$ с не более чем 6 точками разбиения границы в случае, когда триангуляции с границы продолжаются на внутренность симплекса без добавления новых точек разбиения. В результате получены группы гомологий H_n для $n = 0, \dots, 5$.

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