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Asymptotic Formula for the Moments of Bernoulli Convolutions

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Abstract.
For each λ, $0 < \lambda < 1$, we define a random variable

$$Y_\lambda = (1-\lambda)\sum_{n=0}^{\infty} \xi_n \lambda^n,$$

where ξ_n are independent random variables with

$$\mathsf{P}\{\xi_n = 0\} = \mathsf{P}\{\xi_n = 1\} = \frac{1}{2}.$$

The distribution of Y_λ is called a symmetric Bernoulli convolution. The main result of this paper is

$$M_n = \mathsf{E}Y_\lambda^n = n^{\log_\lambda 2} 2^{\log_\lambda(1-\lambda)+0.5\log_\lambda 2-0.5} e^{\tau(-\log_\lambda n)} \left(1 + \mathcal{O}(n^{-0.99})\right),$$

where

$$\tau(x) = \sum_{k\neq 0} \frac{1}{k}\alpha\left(-\frac{k}{\ln\lambda}\right) e^{2\pi i k x}$$

is a 1-periodic function,

$$\alpha(t) = -\frac{1}{2i\,\mathrm{sh}(\pi^2 t)}(1-\lambda)^{2\pi i t}(1-2^{2\pi i t})\pi^{-2\pi i t}2^{-2\pi i t}\zeta(2\pi i t),$$

and $\zeta(z)$ is the Riemann zeta function.

The article is published in the author's wording.

Keywords: moments, self-similar, Bernoulli convolution, singular, Mellin transform, asymptotic

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For each λ, $0 < \lambda < 1$ we define the random variable

$$Y_\lambda = (1-\lambda)\sum_{n=0}^{\infty} \xi_n \lambda^n,$$

where ξ_n are independent random variables with

$$\mathsf{P}\{\xi_n = 0\} = \mathsf{P}\{\xi_n = 1\} = \frac{1}{2}.$$

The distribution of Y_λ is called a symmetric Bernoulli convolution.

The cumulative distribution function $F_\lambda(t) = \mathsf{P}\{Y_\lambda < t\}$ can be characterized by the functional equation

$$F_\lambda(x) = \frac{1}{2}F_\lambda(\lambda^{-1}x) + \frac{1}{2}F_\lambda(\lambda^{-1}x - \lambda^{-1} + 1), \quad 0 \le x \le 1. \tag{1}$$

We stress that the Fourier transform is the infinite product

$$\phi(t) = \mathsf{E}e^{itY_\lambda} = \prod_{n=0}^{\infty}\left(\frac{1}{2} + \frac{1}{2}e^{it\lambda^n(1-\lambda)}\right) = e^{it/2}\prod_{n=0}^{\infty}\cos(\lambda^n(1-\lambda)t/2), \tag{2}$$

The early study of $F_\lambda(t)$ was related to some questions of harmonic analysis [1, 14.20].

In Figure 1 we show the histograms for $F'_\lambda(t)$ approximations. This graphics were created as Iterated Functions System of the maps $S_1(x) = \lambda x$, $S_2(x) = \lambda x - \lambda + 1$. We used 2^{20} points and 1000 equally intervals for the histogram.

We stress that these approximations are very crude.

Since the 1930's a lot of work has been done to investigate $F_\lambda(t)$ (see e.g. survey [8]).

One of the fundamental question is to decide for which λ the function $F_\lambda(t)$ is absolutely continuous and for which it is singular.

Results on absolute continuity of $F_\lambda(t)$.

- Jessen and Wintner [7] proved that $F_\lambda(t)$ is either absolutely continuous or purely singular. It is clear that $F_\lambda(t)$ is uniform on $[0,1]$ for $\lambda = \frac{1}{2}$ and is purely singular for $\lambda < \frac{1}{2}$.
- Wintner [13] proved that $F_\lambda(t)$ is absolutely continuous for $\lambda = 2^{-1/k}$, $k = 1, 2, \dots$, with a density having $k-1$ derivatives. For example, for $\lambda = 2^{-1/2}$ we have the following density

$$F'_\lambda(x) = \begin{cases} \left(2 + \frac{3}{\sqrt{2}}\right)x, & 0 \le x \le \sqrt{2}-1; \\ 1 + \frac{1}{\sqrt{2}}, & \sqrt{2}-1 \le x \le 2-\sqrt{2}; \\ \left(2 + \frac{3}{\sqrt{2}}\right)(1-x), & 2-\sqrt{2} \le x \le 1. \end{cases} \tag{3}$$

- Erdös [4] showed that $F_\lambda(t)$ is singular when λ^{-1} is a Pisot (Pisot-Vijayaraghavan) number ($0.5 < \lambda < 1$). Moreover, the Fourier transform $\phi(t)$ does not tend to 0 as $t \to \infty$. Recall that a Pisot number is an algebraic integer $\theta > 1$ all of whose Galois conjugates (other roots of the minimal polynomial) of θ are less than 1 in modulus.

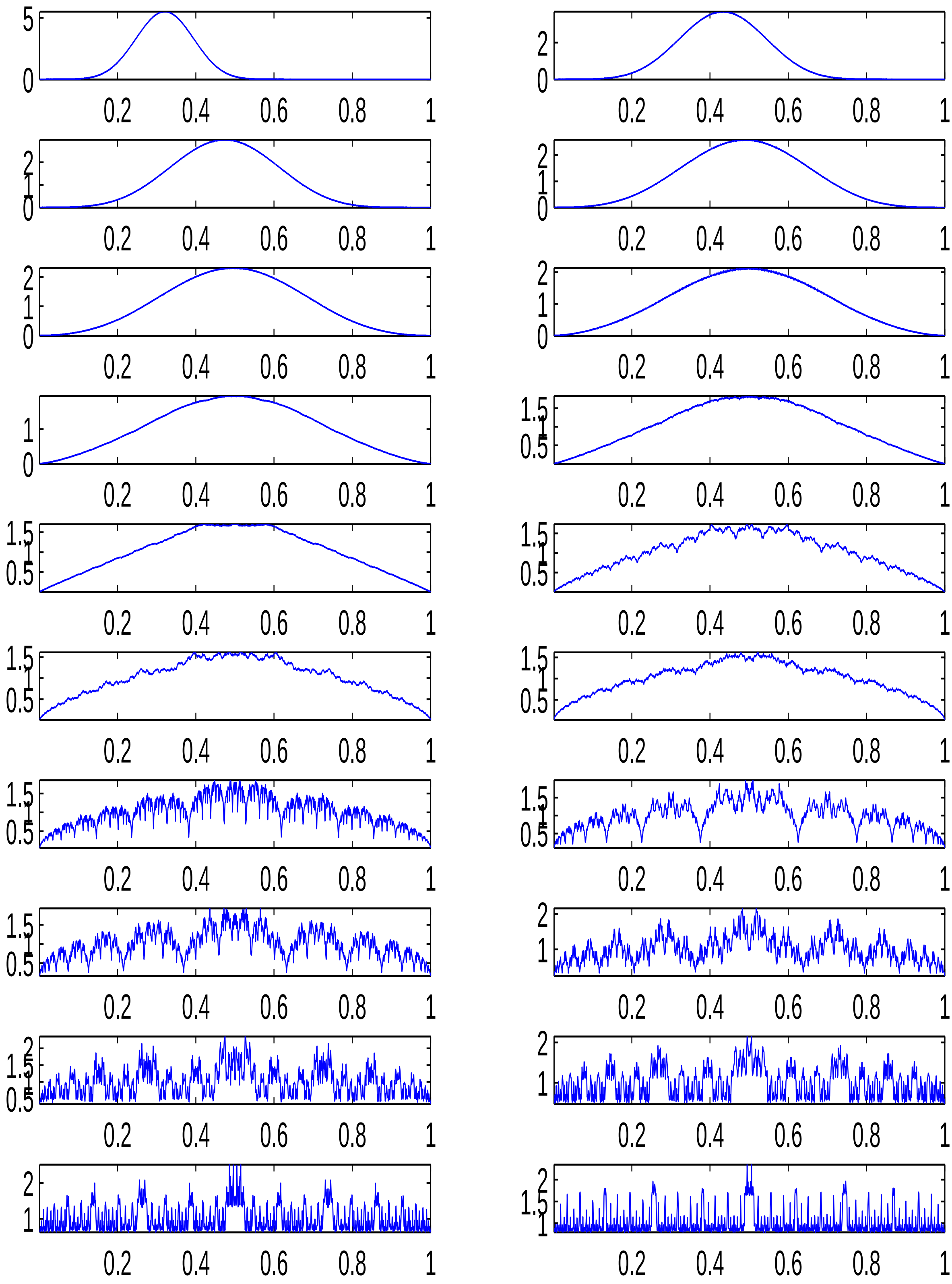

Fig 1. Histograms for $F'_\lambda(t)$ correspond to the following $\lambda^{-1} = 1 + i/21$, $i = 1, 2, \ldots, 20$

- Salem [9–11] proved the converse of the result of Erdos. If $\phi(t)$ does not tend to 0 as $t \to \infty$, then λ^{-1} is necessarily a Pisot number. The reciprocal of Pisot numbers remain today the only known set of λ for which $F_\lambda(t)$ is singular.
- At the next year Erdös [5] proved a result in opposite direction, namely, there is a $a < 1$ such that for almost all λ in the interval $a < \lambda < 1$ we have $F_\lambda(t)$ is absolutely continuous.
- Garsia [6] found new examples of algebraic λ with absolutely continuous $F_\lambda(t)$.
- Solomyak [12] proved that $F_\lambda(t)$ is absolutely continuous with a density in L^2 for a.e. $0.5 < \lambda < 1$.

In this paper we study the moments of Bernoulli convolution. They are defined by

$$M_n = \mathsf{E} Y_\lambda^n = \int_0^1 x^n \, dF_\lambda(x). \tag{4}$$

The main result of this paper

Theorem 1. *Let M_n be defined by (4), then the following holds as $n \to \infty$*

$$M_n = n^{\log_\lambda 2} 2^{\log_\lambda (1-\lambda) + 0.5 \log_\lambda 2 - 0.5} e^{\tau(-\log_\lambda n)} \left(1 + \mathcal{O}(n^{-0.99})\right),$$

where

$$\tau(x) = \sum_{k \neq 0} \frac{1}{k} \alpha \left(-\frac{k}{\ln \lambda}\right) e^{2\pi i k x} \tag{5}$$

is 1-periodic function,

$$\alpha(t) = -\frac{1}{2i \operatorname{sh}(\pi^2 t)} (1-\lambda)^{2\pi i t} (1 - 2^{2\pi i t}) \pi^{-2\pi i t} 2^{-2\pi i t} \zeta(2\pi i t), \tag{6}$$

and $\zeta(z)$ is the Riemann zeta function.

Remark 1. *We emphasize that the periodic function $\tau(x)$ is a constant only in the Wintner's cases $\lambda = 2^{-1/k}$, $k = 1, 2, \ldots$.*

Moreover, only in these cases $F_\lambda(x) = Cx^\alpha + \mathcal{O}(x^{\alpha+\varepsilon})$ for some $\alpha > 0$, $\varepsilon > 0$ as $x \to 0$.

We observe that $|\alpha(t)|$ does not depend on λ, $|\alpha(\pm 1/\ln 2)| \approx 10^{-5}$, and due the fast decrease of $|\alpha(t)|$ (see Figure 2) the fluctuating function $\tau(x)$ stays bounded by 10^{-5}.

Remark 2. *We have the following approximations*

$$M_n \approx n^{\log_\lambda 2} 2^{\log_\lambda (1-\lambda) + 0.5 \log_\lambda 2 - 0.5}$$

with accuracy 10^{-5} and

$$M_n \approx n^{\log_\lambda 2} 2^{\log_\lambda (1-\lambda) + 0.5 \log_\lambda 2 - 0.5} + \alpha \left(-\frac{1}{\ln \lambda}\right) n^{-2\pi i / \ln \lambda} - \alpha \left(\frac{1}{\ln \lambda}\right) e^{+2\pi i / \ln \lambda}$$

with accuracy 10^{-9}.

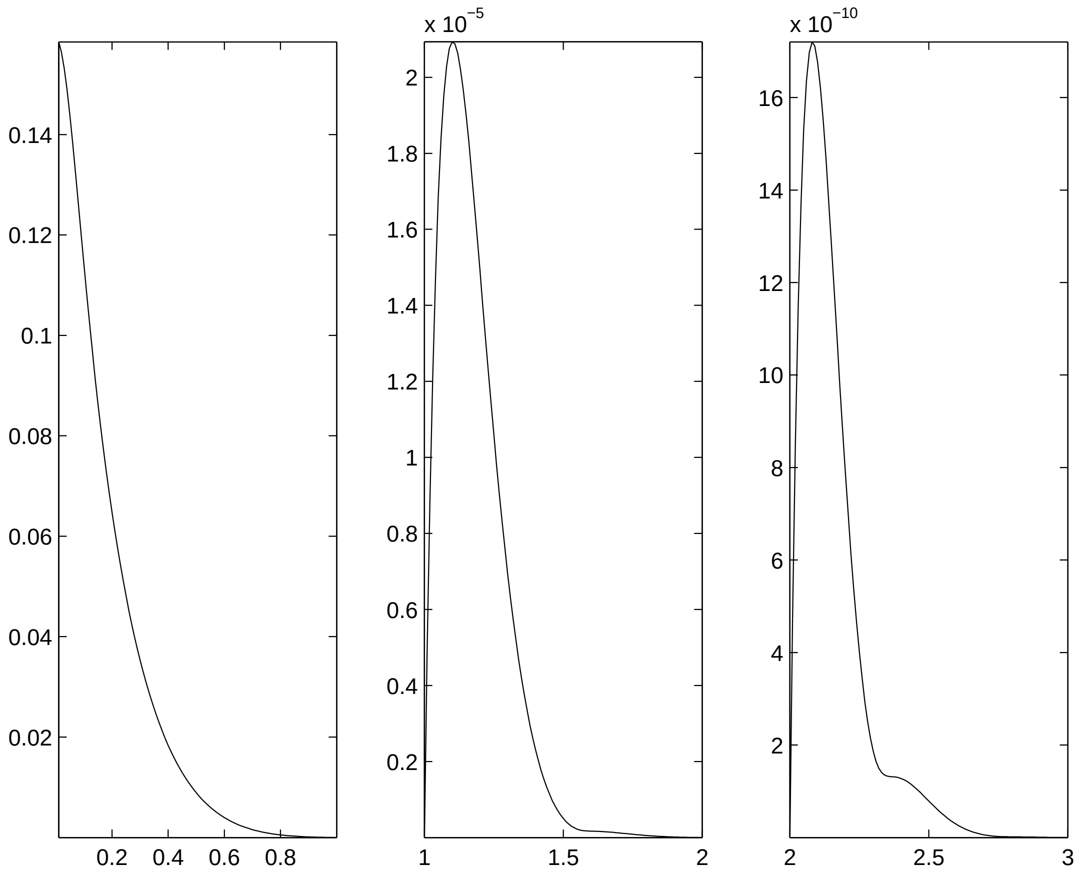

Fig 2. The graphs of $|\alpha(t)|$ for $t \in [0,1]$, $t \in [1,2]$, and $t \in [2,3]$

Proof. Now we give a proof of the theorem using analytic techniques such as poissonization and the Mellin transform

A recursive relation for the moments M_n of any function satisfying (1) is the following

$$M_n = \int_0^1 x^n \, dF_\lambda(x) = \\ = \frac{1}{2}\int_0^1 x^n \, dF_\lambda(\lambda^{-1}x) + \frac{1}{2}\int_0^1 x^n \, dF_\lambda(\lambda^{-1}x - \lambda^{-1} + 1) = \\ = \frac{1}{2}\lambda^n \int_0^1 t^n \, dF_\lambda(t) + \frac{1}{2}\int_0^1 (\lambda t + 1 - \lambda)^n \, dF_\lambda(t).$$

Substituting M_n, we obtain

$$M_n = \frac{1}{2}\lambda^n M_n + \frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}\lambda^k(1-\lambda)^{n-k}M_k, \quad M_0 = 1, \quad n = 0, 1, \ldots. \tag{7}$$

We define the Poisson transform as

$$M(x) = \sum_{n=0}^{\infty} M_n \frac{x^n}{n!} e^{-x}. \tag{8}$$

which exists for all complex x since the series converges due to the estimate $M_m \leq 1$. Substituting (7) в (8), we get

$$M(x) = \\ = \frac{1}{2}\sum_{n=0}^{\infty}\lambda^n M_n \frac{x^n}{n!}e^{-x} + \frac{1}{2}\sum_{n=0}^{\infty}\frac{x^n}{n!}e^{-x}\sum_{k=0}^{n}\binom{n}{k}\lambda^k(1-\lambda)^{n-k}M_k = \\ = \frac{1}{2}M(\lambda x)e^{-(1-\lambda)x} + \frac{1}{2}M(\lambda x). \quad (9)$$

Since $M_0 = 1$, we find that

$$M(x) = \prod_{k=0}^{\infty}\left(\frac{1}{2} + \frac{1}{2}e^{-(1-\lambda)\lambda^k x}\right). \quad (10)$$

Let

$$G(x) = \ln M(x) = \sum_{k=0}^{\infty}\ln\left(\frac{1}{2} + \frac{1}{2}e^{-(1-\lambda)\lambda^k x}\right). \quad (11)$$

The Mellin transform

$$\tilde{G}(z) = \int_0^{\infty} G(x)x^{z-1}dx, \quad (12)$$

is define for

$$-1 < \Re z < 0$$

and satisfies

$$\tilde{G}(z) = \frac{(1-\lambda)^{-z}}{1-\lambda^{-z}}\int_0^{\infty} x^{z-1}\ln\left(\frac{1}{2} + \frac{1}{2}e^{-x}\right)\,dx,$$

By integration by parts, we obtain

$$\tilde{G}(z) = \frac{(1-\lambda)^{-z}}{1-\lambda^{-z}}\frac{1}{z}\int_0^{\infty}\frac{x^z}{1+e^x}\,dx,$$

Using [15, 3.411.3], we get

$$\tilde{G}(z) = \frac{(1-\lambda)^{-z}}{1-\lambda^{-z}}(1-2^{-z})\Gamma(z)\zeta(z+1), \quad (13)$$

for $-1 < \Re z < 0$.

It is known that the Gamma function and the zeta function are both continuable to the complex plane:

- $\Gamma(z)$ has simple poles at the non-positive integers;
- $\zeta(z)$ has only simple pole at $z = 1$.

Therefore, the function $\tilde{G}(z)$ continuable to the complex plane.

$\tilde{G}(z)$ has a double pole at $z = 0$ and simple poles:

- at the non-positive integers, from $\Gamma(z)$,
- at $z = z_k$, from $(1-\lambda^{-z})^{-1}$, where

$$z_k = \frac{2\pi i k}{\ln\lambda}, \quad k \neq 0; \tag{14}$$

Via the inverse Mellin transform we find that

$$G(x) = \frac{1}{2\pi i} \int_{-\sigma-i\infty}^{-\sigma+i\infty} \tilde{G}(z) x^{-z}\, dz, \tag{15}$$

where $0 < \sigma < 1$.

In order to find $G(x)$, we apply the residue theorem for the right half-plane $\Re z > -\sigma$.

Now we find residues of the function $\tilde{G}(z)x^{-z}$.

Using the following formula

$$\operatorname{Res}\left(\frac{P(z)}{zQ(z)}, 0\right) = \frac{P'(0)}{Q'(0)} - \frac{P(0)Q''(0)}{2Q'(0)^2},$$

where

$$P(z) = \frac{1-2^{-z}}{z}\Gamma(z+1)(1-\lambda)^{-z}x^{-z}(z\zeta(z+1)),$$
$$Q(z) = 1-\lambda^{-z},$$

we obtain

$$\operatorname{Res}\left(\tilde{G}(z)x^{-z}, 0\right) = \frac{P'(0)}{Q'(0)} - \frac{P(0)Q''(0)}{2Q'(0)^2}.$$

(There $\operatorname{Res}(f(z), z_0)$ denotes the residue of $f(z)$ at the point $z = z_0$.)

Substituting $(z\zeta(z+1))_{z=0} = 1$, $(z\zeta(z+1))'_{z=0} = -\Gamma'(1)$ [15, 9.536], we get

$$P(0) = \ln 2, \quad P'(0) = -\ln x \ln 2 - \ln(1-\lambda)\ln 2 - \frac{1}{2}\ln^2 2,$$
$$Q'(0) = \ln\lambda, \quad Q''(0) = -\ln^2\lambda.$$

Therefore,

$$\operatorname{Res}\left(\tilde{G}(z)x^{-z}, 0\right) = -\frac{\ln 2}{\ln\lambda}\left(\ln x + \ln(1-\lambda) + \frac{1}{2}\ln 2\right) + \frac{1}{2}\ln 2.$$

The residue at z_k, $k = \pm1, \pm2, \dots$ is

$$\operatorname{Res}\left(\tilde{G}(z)x^{-z}, z_k\right) = \frac{(1-\lambda)^{-z_k}}{\ln\lambda}(1-2^{-z_k})\Gamma(z_k)\zeta(z_k+1)x^{-z_k}.$$

$\Gamma(z)$ decreases exponentially fast along vertical lines while $\zeta(z)$ is only polinomial growth as $\Im z \to \pm\infty$. Thus, Collorary 1 to Theorem 4 from [3] applies here and we have

$$G(x) = \frac{\ln 2}{\ln\lambda}\left(\ln x + \ln(1-\lambda) + \frac{1}{2}\ln 2\right) - \frac{1}{2}\ln 2 - \\ - \frac{1}{\ln\lambda}\sum_{k\neq0}(1-\lambda)^{-z_k}(1-2^{-z_k})\Gamma(z_k)\zeta(z_k+1)x^{-z_k} + \mathcal{O}(x^{-\gamma}), \tag{16}$$

for every $\gamma > 0$.

Take $\gamma = \log_\lambda 2 + 2$.

Using (16) and (11), we get

$$M(x) = x^{\log_\lambda 2} e^{\tau(\log_\lambda x)} \left(1 + \mathcal{O}(x^{-\gamma})\right),$$

where the function τ is defined in (5).

In order to find M_n, we apply Theorem 10.5 from [14]. To apply Theorem 10.5 from [14], we must check that the conditions required in this theorem are actually satisfied. In particular:

there exist β, $0 < \theta < \pi/2$, and $0 < \eta < 1$ such that the following conditions hold for sufficiently large $|z|$, $z = x + iy$:

- for $z \in S_\theta$

$$\left| \frac{1}{2} + \frac{1}{2} e^{-(1-\lambda)z} \right| \lambda^\beta \leq 1 - \eta;$$

- for $z \notin S_\theta$ and some $\alpha < 1$

$$\left| \frac{1}{2} + \frac{1}{2} e^{-(1-\lambda)z} \right| e^{(1-\lambda)x} \leq e^{\alpha(1-\lambda)|z|};$$

where $S_\theta = \{z \ : \ |\Im z| \leq \theta \Re z\}$.

It can easily be checked that this condition holds for every $\beta > \log_\lambda 2$ and we apply Theorem 10.5 from [14] to yeild

$$M_n = M(n) \left(1 + \mathcal{O}(n^{-0.99})\right).$$

for $\beta = \log_\lambda 2 + 0.01$.

The last step consists of simplifying the function $\alpha(t)$ (6). Using (16), we have

$$\alpha(t) = -\frac{1}{2\pi i}(1-\lambda)^{2\pi i t}(1 - 2^{2\pi i t})\Gamma\left(-2\pi i t + 1\right)\zeta\left(-2\pi i t + 1\right).$$

Applying the functional equation of the Reimann zeta function [15, 9.535.3]

$$\Gamma(z)\zeta(z)\cos\frac{\pi z}{2} = \pi^z 2^{z-1}\zeta(1-z),$$

we obtain (6).

□

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Аннотация. Для каждого λ, $0 < \lambda < 1$ определим случайную величину (симметричную свертку Бернулли)

$$Y_\lambda = (1-\lambda)\sum_{n=0}^{\infty} \xi_n \lambda^n,$$

где ξ_n – независимые случайные величины с

$$\mathsf{P}\{\xi_n = 0\} = \mathsf{P}\{\xi_n = 1\} = \frac{1}{2}.$$

Основной результат настоящей работы

$$M_n = \mathsf{E}Y_\lambda^n = n^{\log_\lambda 2} 2^{\log_\lambda(1-\lambda)+0.5\log_\lambda 2 - 0.5} e^{\tau(-\log_\lambda n)} \left(1 + \mathcal{O}(n^{-0.99})\right),$$

где функция

$$\tau(x) = \sum_{k\neq 0} \frac{1}{k} \alpha\left(-\frac{k}{\ln\lambda}\right) e^{2\pi i k x}$$

является периодической с периодом равным 1,

$$\alpha(t) = -\frac{1}{2i\,\mathrm{sh}(\pi^2 t)}(1-\lambda)^{2\pi i t}(1-2^{2\pi i t})\pi^{-2\pi i t}2^{-2\pi i t}\zeta(2\pi i t),$$

а $\zeta(z)$ – дзета-функция Римана.

Статья публикуется в авторской редакции.

Ключевые слова: моменты, самоподобие, свертка Бернулли, сингулярная функция, преобразование Меллина, асимптотика

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