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Universal Hypergraphic Automata Representation by Autonomous Input Symbols

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Abstract. Hypergraphic automata are automata with state sets and input symbol sets being hypergraphs which are invariant under actions of transition and output functions. Universally attracting objects of a category of hypergraphic automata are automata $\mathrm{Atm}(H_1, H_2)$. Here, H_1 is a state hypergraph, H_2 is classified as an output symbol hypergraph, and $S = \mathrm{End}\, H_1 \times \mathrm{Hom}(H_1, H_2)$ is an input symbol semigroup. Such automata are called universal hypergraphic automata. The input symbol semigroup S of such an automaton $\mathrm{Atm}(H_1, H_2)$ is an algebra of mappings for such an automaton. Semigroup properties are interconnected with properties of the algebraic structure of the automaton. Thus, we can study universal hypergraphic automata with the help of their input symbol semigroups. In this paper, we investigated a representation problem of universal hypergraphic automata in their input symbol semigroup. The main result of the current study describes a universal hypergraphic automaton as a multiple-set algebraic structure canonically constructed from autonomous input automaton symbols. Such a structure is one of the major tools for proving relatively elementary definability of considered universal hypergraphic automata in a class of semigroups in order to analyze interrelation of elementary characteristics of universal hypergraphic automata and their input symbol semigroups. The main result of the paper is the solution of this problem for universal hypergraphic automata for effective hypergraphs with p-definable edges. It is an important class of automata because such an algebraic structure variety includes automata with state sets and output symbol sets represented by projective or affine planes, along with automata with state sets and output symbol sets divided into equivalence classes. The article is published in the authors' wording.

Keywords: automaton, semigroup, hypergraph, input symbol

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Introduction

Automata theory is among major computer science branches studying data conversion devices. Such devices arise in control theory tasks, communication theory problems, economic logistics tasks, and others. A mathematical model of data conversion device is automaton $A = (X, S, Y, \delta, \lambda)$. It is a multiple-set algebra consisting of three sets X, S, Y and two binary operations $\delta : X \times S \to X,\ \ \lambda : X \times S \to Y$. Here X is called a state set, S is classified as an input symbol set, Y is an output symbol set, δ is a transition function, and λ is an output function. The transition function δ for each input symbol $s \in S$ defines the state $\delta(x, s)$, in which the automaton moves from the state $x \in X$ depending on the symbol s. Similarly, the output function λ for each input symbol $s \in S$ determines the output symbol $\lambda(x, s)$. The symbol $\lambda(x, s)$ is generated by the automaton in the state $x \in X$ depending on the symbol s. Thus, for each fixed input symbol $s \in S$ the automaton A determines the transition function $\delta_s : X \to X$ and an output function $\lambda_s : X \to Y$ by the formulas: $\delta_s(x) = \delta(x, s)$ and $\lambda_s(x) = \lambda(x, s)$. For the elements $s, t \in S$, sequential action of transition functions δ_s, δ_t defines associative composition of input symbols $s \cdot t$ so that $\delta_{s \cdot t} = \delta_s \delta_t$.

Therefore, it is often presumed that the input symbol set S is a semigroup interrelated with the transition function and output function of the automaton A by the following formulas: $\delta(x, s \cdot t) = \delta\left(\delta(x, s), t\right),\ \lambda(x, s \cdot t) = \lambda\left(\delta(x, s), t\right)$ for any $x \in X,\ \ s, t \in S$. We also denote the semigroup S as $\mathrm{Inp}(A)$.

Depending on the specifics of the computer science tasks, we can model data conversion device as structured automaton. Its state set X and output symbols set Y are algebraic structures which are invariant under actions of transition and output functions of such automaton. Examples of such structures include a probability space structure, a linear space structure, a topological space structure, an ordered set structure, etc. (see e.g. [1]). Thus, well-known specific computer science tasks lead to the notions of a probability automaton, a linear automaton, a topological automaton, and an ordered automaton. Many authors studied such automata (e.g., [1], [2], [3], [4]). In this approach structured automaton is a focus of scientific interest and current studies of algebraic automata theory, which is an important universal algebra branch. Also, it has a variety of applications to combinatorial automata investigations connected with automaton behavior, analysis and synthesis of automata, as well as to formal language theory, algorithm theory, and many other computer science branches [1], [5].

In the paper we continued to study this field. We investigated algebraic properties of hypergraph automata, i.e. automata with state sets and input symbol sets being hypergraphs [6]. Automata under our study form a wide and important class of automata because a hypergraph is a generalization of such concepts as graph, set partition, plane [7] and others. Thus, such algebraic structure variety includes automata with state sets and output symbol sets represented by planes, along with automata with state sets and output symbol sets divided into equivalence classes.

The main focus of our research is universal hypergraphic automata. Their subautomata cover all homomorphic images of hypergraphic automata (Theorem 1). Such universal automaton for any hypergraphs H_1 and H_2 is the automaton $\mathrm{Atm}(H_1, H_2) = (H_1, S, H_2, \delta^\circ, \lambda^\circ)$, where S is the input symbol semigroup consisting of all pairs $s = (\varphi, \psi)$ of endomorphisms φ of the hypergraph H_1 and homomorphisms ψ from the hypergraph H_1 to the hypergraph H_2, $\delta^\circ(x, s) = \varphi(x)$ is the transition function and $\lambda^\circ(x, s) = \psi(x)$ is the output function (where x is a vertex of H_1 and $s = (\varphi, \psi)$ is an element of S).

According to our previous study, the universal hypergraphic automata are defined up to isomorphism by their input symbol semigroups [8]. Additionally, we solved the problem of concrete characterization of universal automata [9]. The main result of our current study is Theorem 2. It shows the important property of input symbol semigroup of universal hypergraphic automaton which allows to construct an isomorphic copy of the original automaton using input symbol semigroup. Such structure is one of the major tools for proving relatively elementary definability [10] of considered universal hypergraphic automata in a class of semigroups in order to analyze interrelation of elementary characteristics of universal hypergraphic automata and their input symbol semigroups.

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1. Hypergraphic automata

According to A.Bretto [6], a hypergraph is an algebraic system $H = (X, L)$, where X is a nonempty vertex set and L is a family of subsets of the set X called hypergraph edges (or hyperedges). A subset $Y \subseteq X$ is said to be bounded if $Y \subseteq l$ for some $l \in L$, and Y is said to be unbounded otherwise. If hypergraph vertices are incident to some edge, they are called adjacent vertices. The hypergraph is said to be an effective hypergraph if any vertex is incident to some edge of such hypergraph.

Let p be some natural number. The hypergraph H is a hypergraph with p-definable edges if every edge of such hypergraph contains at least $p+1$ vertices and, any p vertices of such hypergraph are incident to no more than one edge.

For example, if we consider planes [7] as hypergraphs with plane points as vertices and plane lines as edges, then any projective plane and any affine plane containing more than 4 points are effective hypergraps with 2-definable edges. Additionally, weak hypergraps studied by A.Molchanov [11] are effective hypergraphs with p-definable edges. Besides, hypergraphs with edges which form partitions of vertex set into equivalence classes containing at least $p+1$ vertices are also effective hypergraphs with p-definable edges.

In additon to such known examples, there are a lot of non-trivial effective hypergraps with p-definable edges for any natural p.

Example 1. *The hypergraph $H = (X, L)$ with the vertex set $X = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and the edges set $L = \{\{1, 2, 3, 4\}, \{1, 5, 6, 7\}, \{1, 2, 5, 8\}\}$ is an effective hypergraph with 3-definable edges(Fig. 1).*

Let $H_1 = (X, L_X)$, $H_2 = (Y, L_Y)$ be any hypergraphs. A homomorphism from H_1 to H_2 is a mapping φ of the set X to the set Y such that adjacent vertices of the hypergraph H_1 are mapped to adjacent vertices of the hypergraph H_2, i.e. the following condition is satisfied

$$(\forall l \in L_X)(\exists l' \in L_Y)(\varphi(l) \subseteq l').$$

Besides, for any $l \in L_Y$ any mapping $\varphi : X \longrightarrow l$ is a homomorphism from H_1 to H_2.

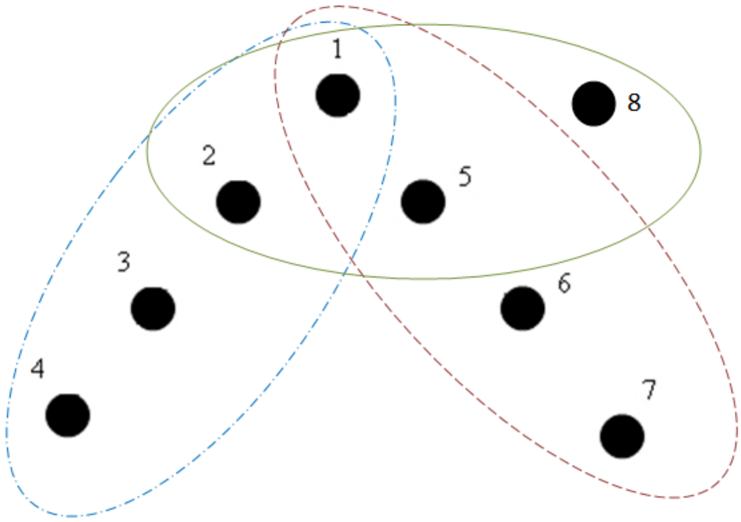

Fig. 1. The effective hypergraph with 3− definable edges H

The set of all homomorphisms from a hypergraph H_1 to a hypergraph H_2 is denoted by $\mathrm{Hom}(H_1, H_2)$. A homomorphism from $H = (X, L)$ to itself is called an endomorphism of H. The set of all endomorphisms of any hypergraph H under the composition operation forms the semigroup $\mathrm{End}\, H$. For hypergraphs $H_1 = (X, L_Y), H_2 = (Y, L_Y)$ by $S(H_1, H_2)$ denote the semigroup $\mathrm{End}\, H_1 \times \mathrm{Hom}(H_1, H_2)$ with binary operation defined by the rule [1]: $(\varphi, \psi)(\varphi_1, \psi_1) = (\varphi\varphi_1, \varphi\psi_1)$ for pairs $(\varphi, \psi), (\varphi_1, \psi_1) \in \mathrm{End}\, H_1 \times \mathrm{Hom}(H_1, H_2)$.

From the algebraic point of view an effective hypergraph with p-definable edges $H = (X, L)$ is an algebraic system $H = (X, L, \rho)$ consisting of two sets X, L and a binary relation $\rho \subset X \times L$, which is defined by the formula $(x, l) \in \rho \iff x \in l$ (where $x \in X, l \in L$) and fulfills the conditions:

$$(\Gamma_1) \quad (\forall x \in X)\, (\exists l \in L)\, ((x, l) \in \rho),$$

$$(\Gamma_2) \quad (\forall l \in L)\, (\exists x_1, x_2, ..., x_{p+1} \in X)(\bigwedge_{1 \le i \ne j \le p+1} x_i \ne x_j \wedge \bigwedge_{1 \le i \le p+1} (x_i, l) \in \rho),$$

$$(\Gamma_3)(\forall x_1, x_2, ..., x_p \in X)(\bigwedge_{1 \le i \ne j \le p} x_i \ne x_j \Rightarrow (\forall l, r)(\bigwedge_{1 \le i \le p} ((x_i, l) \in \rho \wedge (x_i, r) \in \rho)) \Rightarrow$$

$$\Rightarrow r = l)).$$

An isomorphism of such system $H_1 = (X, L_X, \rho)$, $H_2 = (Y, L_Y, \rho')$ is an ordered pair $\pi = (\varphi, \psi)$ of bijections $\varphi : X \to Y$, $\psi : L_X \to L_Y$ preserving system relations, i.e. for any $x \in X, l \in L_X$ the condition $(x, l) \in \rho \iff (\varphi(x), \psi(l)) \in \rho'$ holds.

An automaton $A = (X, S, Y, \delta, \lambda)$ is a hypergraphic automaton if state set X and output symbol set Y are such hypergraphs $H_1 = (X, L_X)$ and $H_2 = (Y, L_Y)$ respectively that for every fixed input symbol $s \in S$ the transformation $\delta_s : X \longrightarrow X$ is an endomorphism of H_1 and the mapping $\lambda_s : X \longrightarrow Y$ is a homomorphism from H_1 to H_2. Such automata is also denoted as $A = (H_1, S, H_2, \delta, \lambda)$.

An input symbol $a \in S$ of automata $A = (X, S, Y, \delta, \lambda)$ is called autonomous if its action is independent of the automaton state, i.e. there is such automaton state, denoted by a_1, and such output automaton symbol, denoted by a_2, that $\delta(x, a) = a_1$, $\lambda(x, a) = a_2$ for all automaton states $x \in X$.

Let $H_1 = (X, L_X), H_2, H_1', H_2'$ be arbitrary hypergraphs, $A = (H_1, S, H_2, \delta, \lambda)$, $A' = (H_1', S', H_2', \delta', \lambda')$ be hypergraphic automata. A homomorphism from A to A' is an ordered triple $\theta = (\pi_1, \gamma, \pi_2)$ of hypergraph homomorphisms $\pi_1 = (\varphi_1, \psi_1) : H_1 \to H_1'$, $\pi_2 = (\varphi_2, \psi_2) : H_2 \to H_2'$ and a semigroup homomorphism $\gamma : S \to S'$, preserving transition functions and output functions of the automata, i.e. the formulas

$$\varphi_1 (\delta(x, s)) = \delta' (\varphi_1(x), \gamma(s)), \quad \varphi_2 (\lambda(x, s)) = \lambda' (\varphi_1(x), \gamma(s))$$

hold for any $x \in X, s \in S$. If π_1, π_2, γ are isomorphisms, then θ is called an isomorphism of hypergraphic automata A and A'.

The important example of hypergraphic automaton is an algebraic system $\mathrm{Atm}(H_1, H_2) = (H_1, S(H_1, H_2), H_2, \delta^\circ, \lambda^\circ)$, where $H_1 = (X, L_X)$, $H_2 = (Y, L_Y)$ are some hypergraphs and for any $x \in X$, $(\varphi, \psi) \in S(H_1, H_2)$ the conditions hold: $\delta^\circ(x, (\varphi, \psi)) = \varphi(x)$, $\lambda^\circ(x, (\varphi, \psi)) = \psi(x)$.

For a set X, let Δ_X denote the identity transformation of X.

Theorem 1. *For any hypergraphic automaton $A = (H_1, S, H_2, \delta, \lambda)$ with state hypergraph $H_1 = (X, L_X)$ and output symbol hypergraph $H_2 = (Y, L_Y)$ there is such homomorphism $\pi : S \longrightarrow S(H_1, H_2)$ that ordered triple $\gamma = (\Delta_X, \pi, \Delta_Y)$ is a homomorphism from the automaton A to the automaton* $\mathrm{Atm}(H_1, H_2)$.

Proof. By definition of hypergraphic automaton $A = (H_1, S, H_2, \delta, \lambda)$, for any $s \in S$ the transformation $\delta_s : X \longrightarrow X$ is an endomorphism of the hypergraph H_1 and the mapping $\lambda_s : X \longrightarrow Y$ is a homomorphism from the hypergraph H_1 to the hypergraph H_2. Thus, $(\delta_s, \lambda_s) \in S(H_1, H_2)$ and we can define a mapping $\pi : S \longrightarrow S(H_1, H_2)$ by the following rule: $\pi(s) = (\delta_s, \lambda_s)$ for every $s \in S$. In accordance with the conditions of interrelation between the input symbol semigroup S and the transition function δ and the output function λ of the automaton A, for every $s, t \in S$ the equalities hold:

$$\pi(s \cdot t) = (\delta_{s \cdot t}, \lambda_{s \cdot t}) = (\delta_s \delta_t, \delta_s \lambda_t) = (\delta_s, \lambda_s)(\delta_t, \lambda_t) = \pi(s)\pi(t).$$

Hence, the mapping π is a homomorphism from the semigroup S to the semigroup $S(H_1, H_2)$. We prove that ordered triple $\gamma = (\Delta_X, \pi, \Delta_Y)$ is a homomorphism from the automaton A to the automaton $\mathrm{Atm}(H_1, H_2)$. It is easy to show that for any state $x \in X$ of the automaton A and any input symbol $s \in S$ of the automaton the equalities hold:

$$\Delta_X(\delta(x, s)) = \delta(x, s) = \delta_s(x) = \delta^\circ(x, \pi(s)) = \delta^\circ(\Delta_X(x), \pi(s)),$$

$$\Delta_Y(\lambda(x, s)) = \lambda(x, s) = \lambda_s(x) = \lambda^\circ(x, \pi(s)) = \lambda^\circ(\Delta_X(x), \pi(s)).$$

Thus, γ is a homomorphism from A to $\mathrm{Atm}(H_1, H_2)$. □

Hence, the automaton $\mathrm{Atm}(H_1, H_2)$ is a universally attracting object [1] of a category of hypergraphic automata with the state hypergraph H_1 and the output symbol hypergraph H_2. Therefore, $\mathrm{Atm}(H_1, H_2)$ is called a universal hypergraphic automaton for the hypergraphs H_1, H_2.

2. Preliminaries

Now we consider the universal hypergraphic automaton $A = \mathrm{Atm}(H_1, H_2)$ for some effective hypergraphs with p-definable edges $H_1 = (X_1, L_1)$ and $H_2 = (X_2, L_2)$. Let C be the set of all autonomous input symbols of the automaton A. Define canonical relations for such automaton:

1) the binary relation ε_1 on C, consisting of such ordered pairs (a, b) of autonomous input symbols $a, b \in C$, which transform states of the automaton A identically, i.e. $(a, b) \in \varepsilon_1 \iff a_1 = b_1$;

2) the binary relation ε_2 on C, consisting of such ordered pairs (a, b) of autonomous input symbols $a, b \in C$, which generate the same output symbols of the automaton A, i.e. $(a, b) \in \varepsilon_2 \iff a_2 = b_2$;

3) the binary relation η_i on C^p, consisting of such ordered pairs (α, β) of elements $\alpha = (a^1, a^2, ..., a^p)$ and $\beta = (b^1, b^2, ..., b^p)$, $a^1, a^2, ..., a^p, b^1, b^2, ..., b^p \in C$, that for every $i = 1, 2$ they map states of A to the bounded set $\{a_i^1, a_i^2, ..., a_i^p, b_i^1, b_i^2, ..., b_i^p\}$ of H_i, i.e.

$$(\alpha, \beta) \in \eta_i \iff \{a_i^1, a_i^2, ..., a_i^p, b_i^1, b_i^2, ..., b_i^p\} \text{ is a bounded set of } H_i \ (i = 1, 2).$$

Let $D_i, i = 1, 2$ denote the set consisting of ordered p-tuples of autonomous input symbols $x^1, x^2, ..., x^p$ such that: $x_i^k \neq x_i^j$ for all $1 \leq k < j \leq p$ and the set $\{x_i^1, x_i^2, ..., x_i^p\}$ is a bounded set of H_i.

Lemma 1. *Let $H_1 = (X_1, L_1)$, $H_2 = (X_2, L_2)$ be effective hypergraps with p-definable edges. Then the canonical relations of the universal hypergraphic automaton $A = \mathrm{Atm}(H_1, H_2)$ satisfy the conditions:*

1) for any state (output symbol, respectively) x of the automaton A there is such autonomous input symbol of the automaton denoted by $\widetilde{x}$ that the automaton A jumps from any state to the state x (outputs symbol x for any state, respectively) due to $\widetilde{x}$, i.e. the condition $\widetilde{x}_1 = x$ holds ($\widetilde{x}_2 = x$, respectively);

2) for each $i = 1, 2$ the relation ε_i is an equivalence relation on the set C and the mapping $\varphi_i : X_i \to C/\varepsilon_i$ defined for $x \in X_i$ by the formula $\varphi_i(x) = \varepsilon_i(\widetilde{x})$ is a bijection from X_i to the factor set C/ε_i;

3) for each $i = 1, 2$ the restriction of η_i to the set D_i is a equivalence relation such that the mapping $\psi_i : L_i \to D_i/\eta_i$ defined for $l \in L_i$ by the formula $\psi_i(l) = \eta_i(\widetilde{x^1}, \widetilde{x^2}, ..., \widetilde{x^p})$ for arbitrary pairwise distinct vertices $x^1, x^2, ..., x^p \in l$ is a bijection from L_i to the factor set D_i/η_i.

Proof. Because the hypergraphs H_1, H_2 are effective, i.e. they satisfy the condition Γ_1, for any state x and any output symbol y of A constant mappings $\varphi : X_1 \to \{x\}$ and $\psi : X_1 \to \{y\}$ are the endomorphism of H_1 and the homomorphism from H_1 to H_2, respectively. Then the pair of mappings $a = (\varphi, \psi)$ is autonomous input symbol of A satisfying the conditions: $a_1 = x, a_2 = y$. Thus, the statement 1) of the lemma is correct.

The statement 2) of the lemma is obviously true.

Consider the restriction of the relation η_1 to the set D_1. Let $\alpha = (a^1, a^2, ..., a^p)$ be an arbitrary element of D_1. According to the definition of the set D_1, $\alpha \in C^p$ and $\{a^1_1, a^2_1, ..., a^p_1\}$ is a bounded set of H_1. Thus, by definition of η_1, the pair $(\alpha, \alpha) \in \eta_1$. Hence, η_1 is a reflexive relation.

For any $\alpha, \beta \in D_1$, where $\alpha = (a^1, a^2, ..., a^p)$, $\beta = (b^1, b^2, ..., b^p)$ for some $a^1, a^2, ..., a^p, b^1, b^2, ..., b^p \in C$, the condition $(\alpha, \beta) \in \eta_1$ means that $\{a^1_1, a^2_1, ..., a^p_1, b^1_1, b^2_1, ..., b^p_1\}$ is a bounded set of H_1. Thus, $\{b^1_1, b^2_1, ..., b^p_1, a^1_1, a^2_1, ..., a^p_1\}$ is also a bounded set of H_1, i.e. $(\beta, \alpha) \in \eta_1$. Hence, η_1 is a symmetric relation.

To prove transitivity of the relation we consider any $\alpha, \beta, \gamma \in D_1$ where $\alpha = (a^1, a^2, ..., a^p)$, $\beta = (b^1, b^2, ..., b^p)$, $\gamma = (c^1, c^2, ..., c^p)$ for some $a^1, a^2, ..., a^p, b^1, b^2, ..., b^p, c^1, c^2, ..., c^p \in C$ satisfying the conditions $(\alpha, \beta), (\beta, \gamma) \in \eta_1$. Thus, the sets $\{a^1_1, a^2_1, ..., a^p_1, b^1_1, b^2_1, ..., b^p_1\}$, $\{b^1_1, b^2_1, ..., b^p_1, c^1_1, c^2_1, ..., c^p_1\}$ are bounded sets of H_1, i.e. there are such edges $l_1, l_2 \in L_1$ that $\{a^1_1, a^2_1, ..., a^p_1, b^1_1, b^2_1, ..., b^p_1\} \subseteq l_1$, $\{b^1_1, b^2_1, ..., b^p_1, c^1_1, c^2_1, ..., c^p_1\} \subseteq l_2$. According to the definition of the set D_1, $a^k_1 \neq a^j_1$, $b^k_1 \neq b^j_1$, $c^k_1 \neq c^j_1$ for all $1 \leq k < j \leq p$. By definition of a hypergraph with p-definable edges, any edge of such hypergraph is uniquely determined by any its distinct p vertices $b^1_1, b^2_1, ..., b^p_1$, i.e. it satisfies the condition Γ_3. Hence, $l_1 = l_2 = l$. Thus, the set $\{a^1_1, a^2_1, ..., a^p_1, c^1_1, c^2_1, ..., c^p_1\} \subseteq l$ is a bounded set of the hypergraph H_1. According to the definition of the relation η_1, we have $(\alpha, \gamma) \in \eta_1$. Hence, the relation η_1 is a transitive relation. Therefore, η_1 is an equivalence relation on D_1, which defines the factor set D_1/η_1.

By definition of a hypergraph with p-definable edges, for each edge $l \in L_1$ there are p distinct vertices $x^1, x^2, ..., x^p \in X_1$ such that $x^1, x^2, ..., x^p \in l$, i.e. the set $\{x^1, x^2, ..., x^p\}$ is a bounded set of H_1. As shown above, there are such input symbols $\widetilde{x^1}, \widetilde{x^2}, ..., \widetilde{x^p}$ of the automaton A that $\widetilde{x^1_1} = x^1, \widetilde{x^2_1} = x^2, ..., \widetilde{x^p_1} = x^p$. As $x^k \neq x^j$, $1 \leq k < j \leq p$, the tuple $\alpha = \left(\widetilde{x^1}, \widetilde{x^2}, ..., \widetilde{x^p}\right)$ is contained in the set D_1 and defines an equivalence class $\eta_1(\alpha)$.

Denote $\psi_1(l) = \eta_1(\widetilde{x^1}, \widetilde{x^2}, ..., \widetilde{x^p})$. As for any p distinct vertices $y^1, y^2, ..., y^p \in l$, autonomous input symbols $\widetilde{x^1}, \widetilde{x^2}, ..., \widetilde{x^p}, \widetilde{y^1}, \widetilde{y^2}, ..., \widetilde{y^p}$ define adjacent vertices $\widetilde{x^1_1} = x^1, \widetilde{x^2_1} = x^2, ..., \widetilde{x^p_1} = x^p, \widetilde{y^1_1} = y^1, \widetilde{y^2_1} = y^2, ..., \widetilde{y^p_1} = y^p$ of H_1 (the set $\{x^1, x^2, ..., x^p, y^1, y^2, ..., y^p\}$ is a bounded set of H_1), the condition $\left(\widetilde{x^1}, \widetilde{x^2}, ..., \widetilde{x^p}\right) \equiv \left(\widetilde{y^1}, \widetilde{y^2}, ..., \widetilde{y^p}\right) (\eta_1)$ holds. Thus, the definition of $\psi_1(l)$ is correct.

Prove that ψ_1 is a bijection from L_1 to the factor set D_1/η_1. For any equivalence class $\eta_1(a^1, a^2, ..., a^p)$ defined by an ordered set $(a^1, a^2, ..., a^p) \in D_1$, we have $a^1, a^2, ..., a^p \in C$, $a^k_1 \neq a^j_1$ for all $1 \leq k < j \leq p$, and the set $\{a^1_1, a^2_1, ..., a^p_1\}$ is a bounded set of H_1. Thus, in H_1 there is such edge $l \in L_1$ that $\{a^1_1, a^2_1, ..., a^p_1\} \subseteq l$. Hence, by definition $\psi_1(l) = \eta_1(a^1, a^2, ..., a^p)$, i.e. the mapping ψ_1 is a surjection from L_1 to the factor set D_1/η_1.

On the other hand, according to the definition of the hypergraph with p-definable edges H_1, for any edges $l, r \in L_1$ satisfying $l \neq r$, there are such vertices $x^1, x^2, ..., x^p, y^1, y^2, ..., y^p \in X_1$ that $x^k \neq x^j$, $y^k \neq y^j$, $1 \leq k < j \leq p$, $x^1, x^2, ..., x^p \in l$, $y^1, y^2, ..., y^p \in r$, and at least one of vertices $y^1, y^2, ..., y^p$ is not contained in l. Then the ordered sets of autonomous input symbols $\alpha = \left(\widetilde{x^1}, \widetilde{x^2}, ..., \widetilde{x^p}\right)$, $\beta = \left(\widetilde{y^1}, \widetilde{y^2}, ..., \widetilde{y^p}\right)$ satisfy the conditions $\alpha, \beta \in D_1$ and $\alpha \not\equiv \beta(\eta_1)$ because vertices $\widetilde{x_1^1} = x^1, \widetilde{x_1^2} = x^2, ..., \widetilde{x_1^p} = x^p$, $\widetilde{y_1^1} = y^1, \widetilde{y_1^2} = y^2, ..., \widetilde{y_1^p} = y^p$ can not belong to the same edge by definiton of the hypergraph with p-definable edges H_1 (the property Γ_3). Hence, the conditions hold: $\psi_1(l) = \eta_1\left(\widetilde{x^1}, \widetilde{x^2}, ..., \widetilde{x^p}\right) = \eta_1(\alpha)$, $\psi_1(r) = \eta_1\left(\widetilde{y^1}, \widetilde{y^2}, ..., \widetilde{y^p}\right) = \eta_1(\beta)$, $\psi_1(l) \neq \psi_1(r)$. Therefore, ψ_1 is one-to-one mapping. Thus, ψ_1 is a bijection from the set L_1 to the factor set D_1/η_1.

It is easy to prove in a similar way that the mapping ψ_2 is a bijection from the set L_2 to the factor set D_2/η_2. Henceforth, the statement 3) of the lemma is true. □

3. Main result

Let $A = \mathrm{Atm}(H_1, H_2)$ be a universal hypergraphic automaton for some effective hypergraphs with p-definable edges $H_1 = (X_1, L_1)$ and $H_2 = (X_2, L_2)$. We introduce the following concepts using the automaton canonical relations:

1) for every $i = 1, 2$ define an algebraic system $\overline{H}_i = \left(\overline{X}_i, \overline{L}_i, \overline{\rho}_i\right)$ with two carrier sets $\overline{X}_i = C/\varepsilon_i$, $\overline{L}_i = D_i/\eta_i$ and a binary relation $\overline{\rho}_i \subset \overline{X}_i \times \overline{L}_i$, which is defined for elements $a, a^1, a^2, ..., a^p \in C$, $a^k \not\equiv a^j(\varepsilon_i), 1 \leq k < j \leq p$ by the formula:
$(\varepsilon_i(a), \eta_i(a^1, a^2, ..., a^p)) \in \overline{\rho}_i \iff \{a_i, a_i^1, a_i^2, ..., a_i^p\}$ is a bounded set of H_i;

2) define two mappings $\overline{\delta} : \overline{X}_1 \times S \to \overline{X}_1$, $\overline{\lambda} : \overline{X}_1 \times S \to \overline{X}_2$ by the formulas for $a \in C, s \in S$:
$$\overline{\delta}\,(\varepsilon_1(a), s) = \varepsilon_1(a \cdot s), \quad \overline{\lambda}\,(\varepsilon_1(a), s) = \varepsilon_2(a \cdot s).$$

Theorem 2. *Let $A = \mathrm{Atm}(H_1, H_2)$ be a universal hypergraphic automaton for some effective hypergraphs with p-definable edges $H_1 = (X_1, L_1)$ and $H_2 = (X_2, L_2)$. Then the following statements are true:*

1) for every $i = 1, 2$ the hypergraph H_i is isomorphic to the algebraic system $\overline{H}_i = \left(\overline{X}_i, \overline{L}_i, \overline{\rho}_i\right)$;

2) the automaton $A = \mathrm{Atm}(H_1, H_2)$ is isomorphic to a hypergraphic automaton $\overline{A} = \left(\overline{H}_1, S, \overline{H}_2, \overline{\delta}, \overline{\lambda}\right)$ with the state hypergraph $\overline{H}_1$, the input symbol semigroup $S = \mathrm{Inp}(A)$, the output symbol hypergraph $\overline{H}_2$, the transition function $\overline{\delta} : \overline{X}_1 \times S \to \overline{X}_1$, and the output function $\overline{\lambda} : \overline{X}_1 \times S \to \overline{X}_2$.

Proof. Consider an algebraic system $\overline{H}_1 = \left(\overline{X}_1, \overline{L}_1, \rho_1\right)$ with two basic sets $\overline{X}_1 = C/\varepsilon_1$, $\overline{L}_1 = D_1/\eta_1$ and the binary relation $\rho_1 \subset \overline{X}_1 \times \overline{L}_1$, which is defined for elements $a, a^1, a^2, ..., a^p \in C$, $a^k \not\equiv a^j(\varepsilon_i)$, $1 \leq k < j \leq p$ by the formula :
$(\varepsilon_1(a), \eta_1(a^1, a^2, ..., a^p)) \in \overline{\rho}_1 \iff \{a_1, a_1^1, a_1^2, ..., a_1^p\}$ is a bounded set of H_1.

According to the statement 2) of Lemma 1, the mapping $\varphi_i : X_1 \to \overline{X}_1$ defined for $x \in X_1$ by the formula $\varphi_1(x) = \varepsilon_1(\widetilde{x})$ is a bijection from the set X_1 to the set $\overline{X}_1$.

In accordance with the statement 3) of Lemma 1, the mapping $\psi_1 : L_1 \to \overline{L}_1$ defined for element $l \in L_1$ by the formula $\psi_1(l) = \eta_1(\widetilde{x^1}, \widetilde{x^2}, ..., \widetilde{x^p})$ for any distinct vertices $x^1, x^2, ..., x^p \in l$ is a bijection from the set L_1 to the set $\overline{L}_1$.

Let a vertex $x \in X_1$ is incident to an edge $l \in L_1$. Then for the hypergraph with p-definable edges H_1 by the property Γ_2, there are at least p distinct vertices $x^1, x^2, ..., x^p \in X_1$ such that $x^1, x^2, ..., x^p \in l$. It was already proved that in the automaton A there are such autonomous input symbols $\widetilde{x^1}, \widetilde{x^2}, ..., \widetilde{x^p}, \widetilde{x}$ that $\widetilde{x^1_1} = x^1, \widetilde{x^2_1} = x^2, ..., \widetilde{x^p_1} = x^p, \widetilde{x}_1 = x$. Then $\varphi_1(x) = \varepsilon_1(\widetilde{x})$, $\psi_1(l) = \eta_1 \left(\widetilde{x^1}, \widetilde{x^2}, ..., \widetilde{x^p}\right)$, and $\{\widetilde{x^1_1}, \widetilde{x^2_1}, ..., \widetilde{x^p_1}, \widetilde{x}_1\} \subseteq l$. Thus, by definition of $\overline{\rho}_1$, the condition $(\varphi_1(x), \psi_1(l)) \in \overline{\rho}_1$ holds.

On the other hand, let $(\varphi_1(x), \psi_1(l)) \in \overline{\rho}_1$ for some elements $x \in X_1$, $l \in L_1$. Then $\varphi_1(x) = \varepsilon_1(\widetilde{x})$, $\psi_1(l) = \eta_1(\widetilde{x^1}, \widetilde{x^2}, ..., \widetilde{x^p})$ for some distinct vertices $x^1, x^2, ..., x^p \in l$ and $\left(\varepsilon_1(\widetilde{x}), \eta_1(\widetilde{x^1}, \widetilde{x^2}, ..., \widetilde{x^p})\right) \in \overline{\rho}_1$. According to the definition of the relation $\overline{\rho}_1$, this condition means that vertices $\widetilde{x^1_1} = x^1, \widetilde{x^2_1} = x^2, ..., \widetilde{x^p_1} = x^p, \widetilde{x}_1 = x$ belongs to the same edge r of the hypergraph H_1. In accordance with definition of an effective hypergraph with p-definable edges, any edge is uniquely determined by any p vertices $x^1, x^2, ..., x^p$ (the condition Γ_3). Thus, $l = r$. Therefore, $x \in l$. Hence, $\pi_1 = (\varphi_1, \psi_1)$ is an isomorphism from the hypergraph H_1 to the algebraic system $\overline{H}_1$.

It is easy to prove in a similar way that the ordered pair of mappings $\pi_2 = (\varphi_2, \psi_2)$ is an isomorphism from the hypergraph H_2 to the algebraic system $\overline{H}_2$. Thus, the statement 1) of the theorem is true.

It follows from 1), that the algebraic systems $\overline{H}_1 = \left(\overline{X}_1, \overline{L}_1, \overline{\rho}_1\right)$, $\overline{H}_2 = \left(\overline{X}_2, \overline{L}_2, \overline{\rho}_2\right)$ are effective hypergraphs with p-definable edges and the algebraic system $\overline{A} = \left(\overline{H}_1, S, \overline{H}_2, \overline{\delta}, \overline{\lambda}\right)$ is a hypergraphic automaton for the hypergraphs H_1, H_2. We denote the ordered triple (π_1, Δ_S, π_2) by θ and prove that θ is an isomorphism from the universal hypergraphic automaton $A = \mathrm{Atm}(H_1, H_2)$ to the hypergraphic automaton $\overline{A}$. It remains to prove that the triple θ preserves transiton functions and output functions of the automata, i.e. for any $x \in X_1, s \in S$ the conditions hold:

$$\varphi_1 (\delta^\circ(x, s)) = \overline{\delta} (\varphi_1(x), s), \quad \varphi_2 (\lambda^\circ(x, s)) = \overline{\lambda} (\varphi_1(x), s).$$

For any automaton state $x \in X_1$ the value $\widetilde{x}$ is an autonomous input symbol such that the automaton jumps to the state x, i.e. the condition $\widetilde{x}_1 = x$ holds. It is easy to see that for any input symbol $s \in S$ the composition $\widetilde{x} \cdot s$ is an autonomous input symbol of the automaton A and the automaton jumps to the state $\delta^\circ(x, s)$ as well as generates the symbol $\lambda^\circ(x, s)$ depending on $\widetilde{x} \cdot s$, i.e. the conditions hold: $(\widetilde{x} \cdot s)_1 = \delta^\circ(x, s)$, $(\widetilde{x} \cdot s)_2 = \lambda^\circ(x, s)$. Thus, by definition of δ°, $\widetilde{\delta^\circ(x, s)} = \widetilde{x} \cdot s$ and the equalities hold:

$$\varphi_1 (\delta^\circ(x, s)) = \varepsilon_1 \left(\widetilde{\delta^\circ(x, s)}\right) = \varepsilon_1 (\widetilde{x} \cdot s) = \overline{\delta} (\varepsilon_1 (\widetilde{x}), s) = \overline{\delta} (\varphi_1(x), s).$$

Similarly, by definition of λ°, $\widetilde{\lambda^\circ(x, s)} = \widetilde{x} \cdot s$ and the equalities hold:

$$\varphi_2 (\lambda^\circ(x, s)) = \varepsilon_2 \left(\widetilde{\lambda^\circ(x, s)}\right) = \varepsilon_2 (\widetilde{x} \cdot s) = \overline{\lambda} (\varepsilon_1 (\widetilde{x}), s) = \overline{\lambda} (\varphi_1(x), s).$$

Hence, $\theta = (\pi_1, \Delta_S, \pi_2)$ is an isomorphism from the universal hypergraphic automaton $A = \mathrm{Atm}(H_1, H_2)$ to the hypergraphic automaton $\overline{A}$. □

4. Conclusions

The main result of the paper allows us to represent a universal hypergraphic automaton for effective hypergraphs with p-definable edges as an algebraic structure canonically constructed in the input symbol semigroup of the automaton. This representation gives us an effective tool for studying a correlation between of properties of universal hypergraphic automata and their input symbol semigroups. The major tools of the representation of a universal hypergraphic automaton in his input symbol semigroup are the canonical relations of the automaton. We plan to prove that these relations are defined by formulas of the first order logic in the input symbols semigroups of the automata.

Based on our approach, we will prove the relatively elementary definability [10] of the class of considered universal hypergraphic automata in the class of all semigroups. It will allow us to investigate the abstract representation problem for universal hypergraphic automata, the elementary definability problem of universal hypergraphic automata by their input symbol semigroup, the algorithmic solvability problem of elementary theories of universal hypergraphic automata, and others.

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Аннотация. Гиперграфическими автоматами называются автоматы, у которых множества состояний и выходных символов наделены структурами гиперграфов, сохраняющимися функциями переходов и выходными функциями. Универсальные притягивающие объекты в категории таких автоматов представляются автоматами $\mathrm{Atm}(H_1, H_2)$ с гиперграфом состояний H_1, гиперграфом выходных символов H_2 и полугруппой входных символов $S = \mathrm{End}\, H_1 \times \mathrm{Hom}(H_1, H_2)$, которые называются универсальными гиперграфическими автоматами. Для такого автомата $\mathrm{Atm}(H_1, H_2)$ полугруппа входных символов S является производной алгеброй отображений, свойства которой взаимосвязаны со свойствами алгебраической структуры данного автомата. Это позволяет изучать универсальные гиперграфические автоматы с помощью исследования их полугрупп входных символов. В настоящей работе рассматривается проблема представления универсальных гиперграфических автоматов в их полугруппах входных сигналов: описывается представление универсального гиперграфического автомата в виде многосортной алгебраической системы, канонически построенной из автономных входных сигналов этого автомата. Эта конструкция является одним из инструментов доказательства относительно элементарной определимости рассматриваемых автоматов в классе полугрупп, которая позволяет проанализировать взаимосвязь элементарных свойств этих автоматов и их полугрупп входных сигналов. Основной результат работы дает решение этой задачи для универсальных гиперграфических автоматов над эффективными гиперграфами с p-определимыми ребрами. Это достаточно широкий и весьма важный класс автоматов, так как он содержит, в частности, автоматы, у которых гиперграфы состояний и выходных символов являются плоскостями (например, проективными или аффинными) или разбиениями на классы нетривиальных эквивалентностей. Статья публикуется в авторской редакции.

Ключевые слова: автомат, полугруппа, гиперграф, входной сигнал

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